What is Market Design?

- Design of auction-based market institutions
  - Buying or selling goods and services
  - Procurement
  - Natural Resources
- Matching two or more “sides” of a market using an auction for allocation
  - Buyers to sellers on eBay
  - Advertisers, consumers and publishers on advertising markets
- Two-sided platform-based markets
  - Two groups of customers, externalities (typically indirect network effects), platform can help internalize externalities
  - Online dating, speed dating
  - Media Markets
- Market design without prices: matching markets
  - Students to schools, classes, sororities
  - New M.D.’s to residency programs
  - Kidneys to donors
Why must markets be designed?

- **Simple Auctions:** Unique objects to sell or procure
  - Timber tracts, oil reserves
  - Defense contracts
  - Spectrum Licenses
  - Search queries

- **Simple Auctions:** Perishable or time-dependent objects
  - Fish
  - Flowers
  - A click on a web site or a search query typed

- **Multi-unit auctions, matching algorithms:** Complex allocation problems
  - Assign advertisers to sponsored link positions for a search phrase
  - Fill a fixed number of positions in a sorority or firm
  - Kidney donations need to match recipient type

- **Decentralized markets don’t work**
  - Market prices don’t exist or don’t convey all relevant information
  - Externalities (indirect network effects)
  - Markets work too slowly
Other Reasons for Using Auctions

- Transparent and objective (governments, encourage long-term complementary investments)
- Efficient auctions promote efficient entry and investment in advance
- Design can be fine-tuned to incorporate public policy goals, such as preferences to particular types of bidders
- Design can be fine-tuned to build critical mass in auction-based marketplaces
- Transparent mechanism facilitates search process (eBay v. Craig’s List)
Auctions and Market Design

• How do market institutions and rules affect:
  – Efficiency
  – Competition
  – Entry, investment, and market structure
  – Ability to collude (information revelation, frequency, ability to respond)

• Competing market institutions
  – Amazon v. eBay
  – Microsoft v. Google
  – Online v. offline

• Competing auctions
  – Different sellers on eBay
  – Forest service, government procurement

• Multi-unit auctions

• Auction-based platforms (eBay, adWords)

• Auctions with resale
Why are auctions interesting to Theorists?

Real-World Problems Motivating Challenging Theoretical Questions, Development of New Methodology and Broadly Applicable Insights

• How are prices formed?

• When do we expect efficient allocation?

• What auction designs achieve various goals?
  – Efficient allocation
  – Robustness to collusion, manipulation
  – Economize on information gathering and processing
  – Economize on computation

• Auctions and the market
  – Auctions aggregate information (financial markets)
  – Auctions determine the conditions for future competition (spectrum)
  – Auctions determine industry structure (timber)
  – Auctions in the center of a 2-sided market (cars, eBay, online advertising)
• Real-world markets more complex than models, challenge to extend our models
  – Tbills
  – Multi-unit auctions
  – Electricity
  – Online advertising
  – Troubled assets

• Beautiful theory, generalizable insights and methods
  – Mechanism design theory is of independent interest to theorists
  – Methods apply to many other problems
  – Auctions are a metaphor for more general markets
Why are auctions interesting to IO Economists?

Auctions as a laboratory for general IO questions

- Theoretical questions, plus empirical applications
- Price formation, efficient allocation
- Should we expect:
  - Efficient entry and investment
  - Market dominance
  - Exercise of market power
- Do firms respond to subtle game-theoretic and information-theoretic incentives in an attempt to exercise market power or in a way that leads to inefficiency?
- Do dynamic considerations affect behavior? Strategically?
  - Is there collusion?
    * If so, what forms does it take?
    * How can you test for it?
    * Do market institutions make it easier or harder?
  - Is there predation?
  - Are there barriers to entry (e.g. informational)?
- Does adverse selection play an important role in this market (lemons market v. winner’s curse)?
  - If so, how do market institutions alleviate or exacerbate the problem?
Auctions and Industry Studies

• Critical importance for specific industries
  – Electricity
  – Timber
    * U.S. Forest Service loses money
    * U.S.-Canada trade dispute
  – Government Procurement
  – Treasury Bills
  – Oil
  – Internet
  – Online advertising
  – Spectrum
  – Privatization
  – Investment Bank
• Approach for industry studies
  
  – Model specific features of auction
  – New theoretical insights, subtle predictions & bidder incentives
  – Test theories
  – Estimate primitives, counterfactuals
  – Examples of questions:
    * What is optimal mechanism? (Discrim v. uniform-price)
    * What should reserve price be?
    * Dynamics issues in reserve price policies
    * Royalties v. lump-sum?
    * Entry fees v. subsidies
    * Release more information about objects?
    * Investment incentives
    * Damages from collusion
Models of Auctions: Private versus Common Value Auctions

- Note: names are misleading, literature often confused
- Framework: Signals \((X_1, \ldots, X_n)\), values \((V_1, \ldots, V_n)\)
  - Bidder \(i\) would receive \(u_i(V_i - t)\) from winning object and paying \(t\)
- Order statistics: \((Y_1, \ldots, Y_{n-1})\) are order statistics from \((X_2, \ldots, X_n)\).
- Assume exchangeability for ease of notation; some results extend
- Private Values
  - \(V_i | X_i \overset{D}{=} V_i | (X_1, \ldots, X_n)\)
    - Example: \(X_i = V_i\).
  - No other bidder has information about how much I would value consuming item
  - Not necessarily independent \(X\)
    - IPV: independent private values, \(V_i = X_i\)
  - Could have uncertainty about my value—but no information in other bidders’ signals
  - Ex: Something I will consume, have no opportunity to buy elsewhere, or else cost/benefit of substitutes are common knowledge
    - Pie at charity auction
– “Common Values” or “General Affiliated Values”

* Complement of Private Values: $V_i | X_i \neq_D V_i | (X_1, .., X_n)$
* $(X_1, .., X_n, V_1, .., V_n)$ are affiliated
  - Mineral Rights: $V_i = V_j, X_i = V + \varepsilon_i$
  - ICV: e.g. $V_i = V = \sum_{i=1}^{n} X_i, X_i$ independent
* Information about a bidder’s value, $V_i$, is spread among bidders
* Factors that induce dispersed information
  - Knowledge of opportunity cost
  - Prospect of future resale
  - Future macroeconomic conditions matter
  - Future industry conditions matter
  - Different forecasts of future state

– Common v. Private: Important differences

* Information can be affiliated in either case
* Different policy predictions
* Identification
• “Standard” Auction Formats
  
  – Open Outcry
    * Unstructured
    * Jump bids, minimum increments
    * When bidding stops, highest wins and pays bid
  
  – English/Ascending/Button
    * Bid level rises continuously
    * Finger on button indicates participation
    * Can’t re-enter
    * When $n - 1$ drop out, last wins and pays final drop-out point
  
  – Second-Price
    * Submit sealed bids
    * Highest bidder wins, pays second-highest bid
  
  – First-Price
    * Highest bidder wins and pays own bid
    * Equivalent to Dutch/descending (but dataset can have at most one bid)
• The PV Case: Strategies
  
  – Ascending Auction
    
    * Dominant strategy to stay in until bid level rises to $V_i$, e.g. $\beta_i(v_i) = v_i$

  – Second-Price Auction
    
    * Dominant strategy to bid value: $\beta_i(v_i) = v_i$
      
      \begin{itemize}
      \item Bid only affects payoffs if it makes you switch between winning and losing
      \item Bidder receives full social surplus she brings to table
      \item Don’t want to lower bid below $v_i$ and lose profit; don’t want to raise bid above $v_i$ and win at a loss
      \end{itemize}

  – Second-Price v. Ascending
    
    * Correlation among types does not matter for strategies—dominant strategy to bid value
      
      * Don’t learn anything relevant during a PV ascending auction, so you can use a “bidder elf” or “proxy bidder”
– First-Price Auction

* Optimization problem:

\[
\max_{b_i} \ (v_i - b_i) \ Pr(\max_{j \neq i} \beta_j(V_j) \leq b_i | V_i = v_i)
\]

* Oligopoly analogy for procurement auction

* Let

\[
G_i(b_i | v_i) = Pr(\max_{j \neq i} \beta_j(V_j) \leq b_i | V_i = v_i)
\]

Then, first-order condition is

\[
\frac{v_i - b_i}{b_i} = \frac{1}{\frac{\partial}{\partial b_i} G_i(b_i | v_i) \cdot \frac{b_i}{G_i(b_i | v_i)}}
\]

or

\[
v_i = b_i + \frac{G_i(b_i | v_i)}{g_i(b_i | v_i)}
\]

* Note: have expressed this in terms of best responses to opponent actions, rather than in terms of primitives
• Mechanism Design Tools: Setup

- $n$ agents, $i = 1, \ldots, n$
- Base case: $u_i(a_i, x_i) - t_i$
  * Types: $x_i \in [x_i, \bar{x}_i]$, joint distn $f$
- Assumption (SCP): $u_i$ differentiable and supermodular, satisfied for $u_i(a_i, x_i) = a_i x_i$
- Offer menu: $a_i(x), t_i(x)$
- $U_i(\hat{x}_i, x_{-i}|x_i) = u_i(a_i(\hat{x}_i, x_{-i}), x_i) - t_i(\hat{x}_i, x_{-i})$
- Interim:
  * $\bar{U}_i(\hat{x}_i|x_i) = E_{X_{-i}}[U_i(\hat{x}_i, X_{-i}|x_i)|X_i = x_i]$
  * $\bar{a}_i(\hat{x}_i|x_i) = E_{X_{-i}}[a_i(\hat{x}_i, X_{-i})|X_i = x_i]$
  * $\bar{t}_i(\hat{x}_i|x_i) = E_{X_{-i}}[t_i(\hat{x}_i, X_{-i})|X_i = x_i]$
- Ex post IC: $U_i(x_i, x_{-i}|x_i) \geq U_i(\hat{x}_i, x_{-i}|x_i) \forall \hat{x}_i, x_i, x_{-i}$
- Bayesian IC: $\bar{U}_i(x_i|x_i) \geq \bar{U}_i(\hat{x}_i|x_i) \forall \hat{x}_i, x_i$
• The single crossing property and local incentives
  
  – Theorem: Assume (SCP): \( a_i(x), t_i(x) \) is Bayesian IC iff (i) \( a_i \) is nondecreasing, and (ii) \( \frac{\partial}{\partial \hat{x}_i} \bar{U}_i(\hat{x}_i|x_i) = 0 \) whereever the derivative exists.

  * Interpretation: local IC implies global IC
  * Intuition: I don’t want to mimic \( x_i + \varepsilon \), and \( x_i + \varepsilon \) doesn’t want to mimic \( x_i + 2\varepsilon \). But \( a_i() \) is nondecreasing, and \( x_i + \varepsilon \) values \( a_i \) relative to \( t_i \) more than \( x_i \), so certainly \( x_i \) does not want to mimic \( x_i + 2\varepsilon \).

  – It is clear that local IC is necessary
  – The content of the result is that it is also sufficient
Mechanism Design Tools: Proof of RET

- Whether or not the single crossing property holds, local IC is necessary
- Key consequence of Bayesian IC:

\[
\bar{U}_i(x_i|x_i) = \bar{U}_i(x_i|x_i) + \int_{\bar{x}_i}^{x_i} \bar{U}_{i,x_i}(\bar{x}|\bar{x}_i)d\bar{x}_i
\]

\[
= \bar{U}_i(\bar{x}_i|x_i) - \int_{x_i}^{\bar{x}_i} \bar{U}_{i,x_i}(\bar{x}_i|\bar{x}_i)d\bar{x}_i
\]

* Difference between utility of two types is “efficiency rent,” extent to which types value \(a\) differently

- Auction setup:

\[
\bar{U}_i(\hat{x}_i|x_i) = \bar{a}_i(\hat{x}_i|x_i)x_i - \bar{t}_i(\hat{x}_i|x_i)
\]

independent values implies

\[
\bar{U}_i(\hat{x}_i|x_i) = \bar{a}_i(\hat{x}_i)x_i - \bar{t}_i(\hat{x}_i)
\]

- Then, Bayesian IC implies

\[
\bar{U}_i(x_i|x_i) = \bar{U}_i(x_i|x_i) + \int_{\bar{x}_i}^{x_i} \bar{a}_i(\bar{x}_i)d\bar{x}_i
\]

proving RET.
• Gaining Intuition for the RET

– Consider a 3-type model, with types \( x_{iL} < x_{iM} < x_{iH} \)

– Suppose that principal offers a menu \( \{(a_{iL}, t_{iL}), (a_{iM}, t_{iM}), (a_{iH}, t_{iH})\} \)

  * Note that equilibrium of decentralized first-price auction game with discrete types has only mixed strategy equilibria; so here we are considering scenario where principal determines three choices and no others are available

– Express the low-type transfer as (e.g., \( K \) is an outside option):

\[
t_{iL} = a_{iL}x_{iL} - K
\]

– Suppose downward local incentive constraints hold with equality. Implies

\[
t_{iM} = t_{iL} + (a_{iM} - a_{iL})x_{iM} \\
= -K + a_{iM}x_{iL} + (a_{iM} - a_{iL})(x_{iM} - x_{iL})
\]

\[
t_{iH} = -K + t_{iM} + (a_{iH} - a_{iM})x_{iH} \\
= -K + a_{iH}x_{iL} + (a_{iM} - a_{iL})(x_{iM} - x_{iL}) \\
+ (a_{iH} - a_{iM})(x_{iH} - x_{iL})
\]

– So, transfer to each type determined by how much each type values the lowest-type’s package, plus the cumulative of the extents to which each successive agent values the allocation difference more than his next-lowest neighbor.

– Note that in discrete type model, downward IC’s don’t necessarily bind, but it helps gain intuition for continuum where they do in the games we consider.
Applying the Revenue Equivalence Theorem

- R.E.T. $\Rightarrow$ transfers are the same in expectation.

Take 2nd price auction (SPA):

$$t_i^2(x) = 1_{x_i > \max_{j \neq i} x_j} \cdot \max_{j \neq i} x_j$$

$$E_{X_{-i}}[t_i^2(x_i, X_{-i})] = \Pr(x_i > \max_{j \neq i} X_j) \cdot E(\max_{j \neq i} X_j \mid x_i > \max_{j \neq i} X_j)$$

$$E_{X_{-i}}[t_i^1(x_i, X_{-i})] = \Pr(x_i > \max_{j \neq i} X_j) \cdot \beta_i^1(x_i)$$

So,

$$\beta_i^1(x_i) = E(\max_{j \neq i} X_j \mid x_i > \max_{j \neq i} X_j)$$
Another example: All-pay auction

\[ t_i^{AP}(x) = \beta_i^{AP}(x_i), \]

so

\[ \beta_i^{AP}(x_i) = E_{X_{-i}}[t_i^2(x_i, X_{-i})] = \Pr(x_i > \max_{j \neq i} X_j) \cdot E(\max_j X_j \mid x_i > \max_{j \neq i} X_j) \]

- Ex: \( X_i \sim U[0, 1] \), N firms
  * Computations from 2nd price
    * For \( x_i \), probability of winning is \( x_i \), expected payment conditional on winning is \( x_i(N - 1)/N \)
  * Optimal bid in 1st price for type \( x_i \) must then be \( x_i(N - 1)/N \).
  * Optimal bid in all-pay auction must be \( x_i^2(N - 1)/N \).
  * As \( N \) gets large, bidder surplus disappears
• Generalizing Symmetric IPV Model
  
  – Reserve prices
    * Usually do better to set a reserve price—analogous to distortions in monopoly or price discrimination models
    * Selling to low types decreases revenue I can extract from high types
    * Types close to $r$ don’t get much surplus (don’t shade their bids much in 1st price), spilling over to higher types
  
  – Risk aversion
    * Bidders more aggressive in FPA, so FPA dominates 2nd/English
    * Intuition:
      * 2nd/English unaffected.
      * FPA: shading your bid is a gamble. Decrease bid by $\varepsilon$; gain $\varepsilon$ when you win, but risk losing $v_i - b_i$

  – Asymmetries
    * Note: existence and computation tricky for all but SPA
    * FPA “typically” dominates SPA/English
    * Intuition: Strong bidder less aggressive, weak bidder more aggressive, so inefficient allocation
    * In FPA, strong bidder worries about chance that weak bidder has unusually high draw
    * Also affects equilibrium entry if entry is costly (Athey/Levin/Seira)
Efficient Mechanisms

• Setup
  
  – In auctions: second price auction is (DIC), use it to construct transfers and utility in a (BIC) mechanism
  
  – The same logic applies to more general context: 2nd price auction could be generalized to Groves mechanism (Vickery/Clark).
    
    * Agent preferences:
      
      \[ u^i(a_i, x_i) - t_i \]
    
    * a project choice, public good.

  – Definition:
    
    Allocation \( \{a^*_i(x)\}_{i=1}^I \) is efficient if and only if, for all \( \tilde{a} \),
    
    \[
    \sum_{i=1}^I u^i(a^*_i(x), x_i) \geq \sum_{i=1}^I u^i(\tilde{a}_i, x_i)
    \]
    
    * Could we implement efficient allocation in dominant strategies?
      
      Answer is given by Vickery-Groves-Clark mechanism.
• Vickrey-Groves-Clark:
  
  - **Theorem:**
    * Let $a^*(x)$ be efficient.
    * Then $(a^*, t^*)$ is DIC if, for $i = 1, \ldots, I$,
      \[ t_i(x) = -\sum_{j \neq i} u^j(a^*_j(x), x_j) + h_i(x_{-i}) \text{ for some } h_i(x_{-i}). \]
  
  - **Proof:**
    \[
    u^i(x_i | x) = u^i(a^*_i(x), x_i) - t_i(x) \\
    = u^i(a^*_i(x), x_i) + \sum_{j \neq i} u^j(a^*_j(x), x_i) - h_i(x_{-i}) \\
    = \sum_j u^j(a^*_j(x), x_j) - h_i(x_{-i})
    \]

  
  Objective of each agent is social objective.
  \[
  u^i(\bar{x}_i | x) = \sum_{j=1}^{I} u^j(a^*_j(\bar{x}_i, x_{-i}), x_j) - h_i(x_{-i})
  \]

  - Can we find $h_i(x_{-i})$ to balance budget:
    \[
    \sum_{i=1}^{I} t_i(x) = 0
    \]

    No, in general.

• Special version: pivotal mechanism (Clark)
- Define:

\[ a^{-i*} = \arg \max \sum_{j \neq i} u^j(a, x_i) \]

Optimal allocation in the world where agent \( i \) does not exist

- Define:

\[ h_i(x_{-i}) = \sum_{j \neq i} u^j(a^{-i*}(x_{-i}), x_j) \]

- Example: 2nd price auction. Assume \( x_i \) is highest. Then:

\[
\sum_{j \neq i} u^j(a^{-i*}(x_{-i}), x_j) = \max_{j \neq i} x_j
\]

\[
t_i(x_i) = -\sum_{j \neq i} u^j(a_j^*(x), x_j) + h_i(x_{-i})
\]

\[
= -0 + \max_{j \neq i} x_j
\]

\[
= \max_{j \neq i} x_j
\]

- Interpretation: if you are pivotal, you pay your externality on the world

- Conclusion:

DIC can implement efficient allocation, but budget is not balanced
• Bayesian implementation:

  – Can get balanced budget by using “expected externality”
    mechanism if there are no IR constraints.

  – How to make transfers add up?

  – Define:
    \[ t_i(x) = -E_{x_{-i}}[\sum_{i \neq j} u^j(a^*(x_i, x_{-i}), x_j)] + h_i(x_{-i}) \]
    \[ g_i(x_i) \]

  – BB:
    \[ \sum_{i=1}^{I} t_i(x) = 0 \Rightarrow \sum_{i=1}^{I} [h_i(x_{-i}) - g_i(x_i)] = 0 \]

  – Let:
    \[ h_i(x_{-i}) = \frac{1}{I - 1} \sum_{i \neq j} g_j(x_j) \]

  – Problem: the participation constraint might be violated. After observing type, type \( x_i \) might want to back out. But, incorporating IR constraints would destroy BB.

  – Note: Athey and Segal (2007) extend this to a dynamic game, show how IR constraints can be relaxed with sufficient patience relative to persistence
Optimal (Revenue-Extracting) Auctions

- $x_0$ principal’s value of the object

  - Principal’s surplus is:

    $$\pi = E\left[ (1 - \sum_{i=1}^{I} a_i(x)) x_0 + \sum_{i=1}^{I} a_i(x) x_i - \sum_{i=1}^{I} \bar{U}^i(x_i | x_i) \right]$$

  - If BIC holds, substitute in indirect utility:

    $$= E \left[ \left( 1 - \sum_{i=1}^{I} a_i(x) \right) x_0 + \sum_{i=1}^{I} a_i(x) x_i \right] - \sum_{i=1}^{I} \left( \bar{U}^i(x_i | x_i) + \int_{x_i}^{x_i} \bar{a}_i(s) \, ds \right)$$

  - Use integration by parts (of $\int_{x_i}^{x_i} \bar{a}_i(s) \, ds \, f(x_i) \, dx_i = -\int \bar{a}_i(x_i) (1 - F_i(x_i)) \, dx_i$), to get

    $$= E \left[ \left( 1 - \sum_{i=1}^{I} a_i(x) \right) x_0 + \sum_{i=1}^{I} a_i(x) x_i \right] \left[ x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right] - \sum_{i=1}^{I} \bar{U}^i(x_i | x_i)$$
Define virtual type: produces surplus, creates incentive problems for higher types

\[ J_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \]

\[ \pi = E \left[ \sum_{i=1}^{I} a_i(x) [J_i(x_i) - x_0] - \sum_{i=1}^{I} U^i(x_i \mid x_i) \right] \]

If IR binds to zero:

\[ U^i(x_i \mid x_i) = 0 \]

Solve:

\[ \max \sum_{i=1}^{I} a_i(x) [J_i(x_i) - x_0] \quad \text{s.t.} \quad \sum_{i=1}^{I} a_i(x) = 1 \]

\[ a_i(x) = \begin{cases} 1 & \text{if } J_i(x_i) > J_k(x_k) \text{ and } J_i(x_i) > x_0 \quad \forall k \neq i \\ 0 & \text{otherwise} \end{cases} \]

* If types are symmetric \( (F_i = F) \) and \( x_i - \frac{1 - F(x_i)}{f(x_i)} \) is \( \not/ \) \( x_i \), then

\[ J_i(x_i) > J_k(x_k) \iff x_i > x_k \]

\( \Rightarrow \) optimal auction entails efficient allocation among bidders.

* If MHC holds, then \( J_i(x_i) \not/ \)

* Reserve prices:

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Optimal rule: sell the object iff:

$$x_i - \frac{1 - F(x_i)}{f(x_i)} > x_0$$

We could have $x_i > x_0$ but $J(x_i) < x_0$, in which case the principal withholds good inefficiently.

- Asymmetric Bidders
  - Bulow and Roberts, JPE, 1988: Auctions as monopoly pricing
  - If $F_i \neq F_k$, then $J_i(x_i) > J_k(x_k)$ is not the same as $x_i > x_k$
  - Bias against strong types.

- If $J_i$ non-monotone: “ironing” (see Fudenberg and Tirole); for revenue-minimizing auction, full pooling (McAfee and McMillan (1992), Athey, Bagwell, and Sanchirico (2004)).
• Affiliation

  - Set order: \( A \succeq_{SSO} B \) if \( x \in A, y \in B \) implies \( \max(x, y) \in A \), \( \min(x, y) \in B \)
    
    * \([a_i, b_i]\) nondecreasing in \(a_i, b_i\)

  - Definition/Characterizations

    * \((X_1, \ldots, X_n)\) are affiliated
    
    * \(E[h(X_{-i})|X_{-i} \in S, X_i = x_i]\) is nondecreasing in \((S, x_i)\) for all \(h\) nondecreasing iff
    
    * the density \(f_X(x)\) is log-supermodular
    
    * IDEA: strong notion of positive correlation. affiliation implies:

    * \((g_1(X_1), \ldots, g_n(X_n))\) is positive correlated for all \(g_i\) non-decreasing

  - If \(n = 2\):

    * \((X_1, X_2)\) affiliated iff
    
    * \(X_2|X_1 = x_1\) satisfies MLRP iff
    
    * \(f_{X_2|X_1}(x_2^H|x_1) / f_{X_2|X_1}(x_2^L|x_1)\) is nondecreasing in \(x_1\) iff
    
    * \(E[h(X_2)|X_1 = x_1]\) is single crossing in \(x_1\) for all \(h\) single crossing iff

    * \(\arg\max_{a_1 \in S} E[u(a_1, X_2)|X_1 = x_1]\) is nondecreasing in \((x_1, S)\) for all \(u\) such that \(\arg\max_{a_1 \in S} u(a_1, x_2)\) is non-decreasing in \((x_2, S)\)

    * IDEA: preserves comparative statics

  - See Milgrom/Weber 82, Athey 2002 for details of these results
• Extend Mechanism Design Tools to Affiliated Private Value Auctions: The Linkage Principle
  
  – Use case: $X_i = V_i$
  
  – Equilibrium Strategies:
    * Second-price auction
      • Bidder 1 bids value $V_i$
      • Payment: $Y_1 = \max(X_2, \ldots, X_n)$
    * Ascending auction
      • Bidder 1 stays in until bid rises above your value, $V_i$
      • Payment: $P^{SPA}$
    * First-price auction
      
      \[
      P^{FPA}(x) = \beta^{FPA}(x)
      = \arg \max_b (x - b) \Pr(\beta^{FPA}(Y_1) \leq b \mid X_1 = x)
      \]

  – Single-crossing property: in symmetric model, $(Y_1, X_1)$ are affiliated, and so expected utility has single crossing property in $(b; x)$
– Indirect utility formulation

* Consider symmetric mechanisms that allocate to highest report
* Report is $\hat{x}$ and true type is $x$
* Let $H(y|x) = \Pr(Y_1 \leq y \mid X_1 = x)$.
* Let $U_{i,2}(\hat{x}|x) = \frac{\partial}{\partial x} U_{i}(\hat{x}|x)$
* Expected payment, conditional on having highest report:
  * $P^{SPA}(\hat{x}, x) = \mathbb{E}[Y_1 \mid Y_1 \leq \hat{x}, X_1 = x]$
  * $P^{FPA}(\hat{x}, x) = \beta^{FPA}(\hat{x})$
* Then:
  $$U_i(\hat{x}|x) = H(\hat{x}|x)(x - P^A(\hat{x}, x)),$$
  and
  $$U_{i,2}(\hat{x}|x) = (x-P^A(\hat{x}, x))\frac{\partial}{\partial x} H(\hat{x}|x)+H(\hat{x}|x) \left( 1 - \frac{\partial}{\partial x} P^A(\hat{x}, x) \right)$$

* $\frac{\partial}{\partial x} P^{SPA}(\hat{x}, x) = \frac{\partial}{\partial x} \mathbb{E}[Y_1 \mid Y_1 \leq \hat{x}, X_1 = x] \geq 0$
* $\frac{\partial}{\partial x} P^{FPA}(\hat{x}) = 0$
* Prices start out the same at $x$; anytime indirect utility functions cross, second-price is flatter
* So, utility increases more slowly in a second-price auction
* This is the linkage principle
  
  · Linking payment to things affiliated with $X_1$ decreases revenue.
  
  · Intuition: greater relationship between truth and payment, easier to deter mimicry
* Compare revenue:
  
  · Bayesian incentive compatibility (and boundary condition) require that:

$$U_i(\hat{x}|x) = U_i(x|x) + \int_{\underline{x}}^{x} U_{i,2}(s|s)ds$$

$$= \int_{\underline{x}}^{x} U_{i,2}(s|s)ds$$

· So, if $U_{i,2}^{FPA}(x|x) > U_{i,2}^{SPA}(x|x)$, $U_{i}^{FPA}(x|x) > U_{i}^{SPA}(x|x)$.

· FPA is better for the bidders, worse for the auctioneer!
Common Value Auctions

– Equilibrium Strategies in 2nd-Price Auction

* Incorrect Conjecture (i): bid \( \mathbb{E}[V_i|X_i = x] \).

  · Why not? Suppose \( Y_1 = x - \varepsilon \), and all opponents follow same strategy
  Note that changing bid is irrelevant for opponent signals much lower or higher
  · Then, bidder 1 wins, and when \( x \) is low, expected value of object is
    \[
    \mathbb{E}[V_1|X_1 = x, Y_1 = x - \varepsilon] < \mathbb{E}[V_i|X_i = x] = \mathbb{E}_{Y_i} [\mathbb{E}_{V_i}[V_i|X_i = x, Y_i]].
    \]

When \( x \) is high, opposite holds.

  · Would pay \( \mathbb{E}[V_i|X_i = x - \varepsilon] \), given that opponent follows same strategy
  · Make a loss when \( x \) is low. Then, bidding \( \mathbb{E}[V_i|X_i = x] - \varepsilon \) improves profits for bidder 1, by causing bidder to lose in states where winning is unprofitable
Incorrect Conjecture (ii): bid $\mathbb{E}[V_1|X_1 = x, Y_1 \leq x]$ to account for the winner’s curse.

- Similar logic applies: Suppose $Y_1 = x + \varepsilon$, and all opponents follow same strategy
- Then, bidder 1 loses. To win, given opponent follows same strategy, need to bid

$$\mathbb{E}[V_1|X_1 = x, Y_1 \leq x + \varepsilon]$$

- If bidder 1 deviated to that bid, expected value of object is (where inequality follows by affiliation)

$$\mathbb{E}[V_1|X_1 = x, Y_1 = x + \varepsilon] > \mathbb{E}[V_1|X_1 = x, Y_1 \leq x],$$

- Would earn positive profits.
- Increasing bid by $\varepsilon$ does not affect payments for far-away realizations of $Y_1$, and increases profits for nearby ones.

Actual strategy:

$$\beta^{SPA}(x) = \mathbb{E}[V_1|X_1 = x, Y_1 = x],$$

which is value of object when opponents have highest possible signal where bidder 1 still wins with $X_1 = x$.

- Following logic from above, if $Y_1 << X_1$ or $Y_1 >> X_1$, changing bid does not affect anything. Only time bid is relevant is if $Y_1$ is close to $X_1$.
- But, when $Y_1 = X_1$, proposed strategy is “bid your value,” whereby bidder 1 is indifferent.
- Note: This is ex post eqm, not dominant strategy.
– Equilibrium Strategies in Ascending Auction

* Use “button auction”
  * Note: this matters, and there are other variants
  * See job market paper by Izmalkov (MIT)

* Let \( k \) denote number of bidders who have quit at current time, and let \( p_1 \leq \cdots \leq p_k \) be prices at which they drop out

* In an equilibrium in strictly increasing strategies, dropping out fully reveals signal

* Recent paper by Bikhchandani, Haile and Riley: continuum of symmetric equilibria in weakly undominated strategies
* Milgrom-Weber equilibrium (highest bids):

\[
\beta^E_0(x) = \mathbb{E}[V_1 | X_1 = x, Y_1 = \cdots = Y_{n-1} = x]
\]

\[
\beta^E_k(x|p_1, \ldots, p_k) = \mathbb{E} \left[ V_1 \left| X_1 = x, Y_1 = \cdots = Y_{n-k-1} = x, \right. \beta^E_{k-1}(Y_{n-k}|p_1, \ldots, p_{k-1}) = p_k, \right. \right. \\
\left. \left. \ldots, \beta^E_0(Y_{n-1}) = p_1 \right) \right]
\]

so that, if \((x_1, \ldots, x_k)\) are signals of first \(k\) bidders to drop out, the bid is equal to

\[
\mathbb{E} \left[ V_1 \left| X_1 = x, Y_1 = \cdots = Y_{n-k-1} = x, \right. Y_{n-k} = x_k, \ldots, Y_{n-1} = x_1 \right].
\]

* Expected value of transaction price given \(x\) is highest and given realizations \(y_1, \ldots, y_{n-1}\):

\[
\mathbb{E} [V_1 | X_1 = Y_1 = y_1, Y_2 = y_2, \ldots, Y_{n-1} = y_n].
\]

* By linkage principle, revenue higher than in 2nd price auction. Price incorporates more pieces of information that are affiliated with true value.
– Equilibrium Strategies in the First-Price Auction

* Solve:

$$\max_b \ (E[V_1|X_1 = x, \beta^{FPA}(Y_1) \leq b] - b)$$

$$\Pr(\beta^{FPA}(Y_1) \leq b|X_1 = x).$$

* Get:

$$\beta^{FPA}(x) = \mathbb{E}[V_1|X_1 = x, Y_1 = x]$$

$$\frac{\partial}{\partial b} \frac{\Pr(\beta^{FPA}(Y_1) \leq b|X_1 = x)}{\Pr(\beta^{FPA}(Y_1) \leq b|X_1 = x)}.$$ 

* Same as private value, just replace $V_1$ with “pseudo-value.”

* Payment depends only on “report,” not any other info affiliated with truth

* So, FPA has lower revenue by linkage principle
Market Design for Competitive, One-Shot Single Unit Auctions with Fixed Participation: Putting the Insights to Work

- Is the auction common value or private value?
- Are the bidders asymmetric?
- Are the bidders sophisticated in understanding the nature of the game?
- Do the bidders face a lot of uncertainty about one another’s valuations?
- How does the designer weigh efficiency, revenue and distribution?
- Is the auction high-stakes?
- Are valuations strategically sensitive to bidders?