A Appendix

A.1 Additional Empirical Results

A.1.1 Estimates of Covariances

Parameter Estimates		
Parameter	Estimate	Std Error
$\sigma_{x\xi} \times 10^{-4}$	0.0004	75.0
$\sigma_{xm} \times 10^{-2}$	-9.36	3.68
$\sigma_{x\pi} \times 10^{-2}$	0.01	15.9
$\sigma_{\lambda m} \times 10^{-4}$	-2.43	0.40
$\sigma_{\xi m} \times 10^{-2}$	-5.08	3.09
$\sigma_{\xi\pi} \times 10^{-2}$	-2.92	-25.3
$\sigma_{m\pi} \times 10^{-2}$	1.09	6.02

A.1.2 Results for Model Freely Estimating σ_{Xm} and $\sigma_{\Lambda m}$

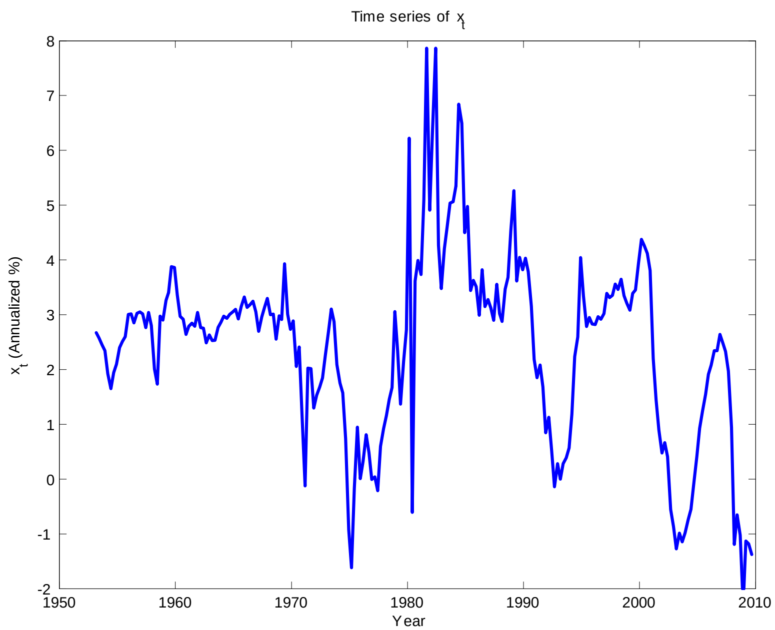

Figure 1: Time series of real rate. The figure on the left plots the time series of x_t, the real interest rate.

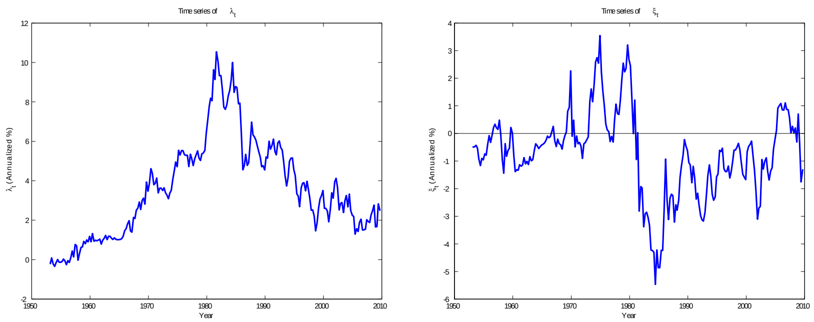

Figure 2: Time series of permanent and transitory components of expected inflation. The figure on the left plots the time series of λ_t, the permanent component of expected inflation. The figure on the right plots the time series of ξ_t, the temporary component of expected inflation.

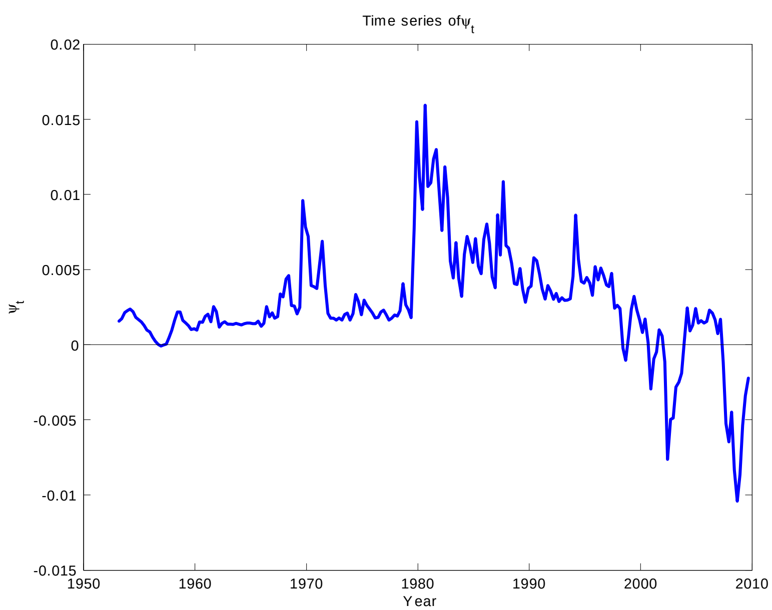

Figure 3: Time series of ψ_t.

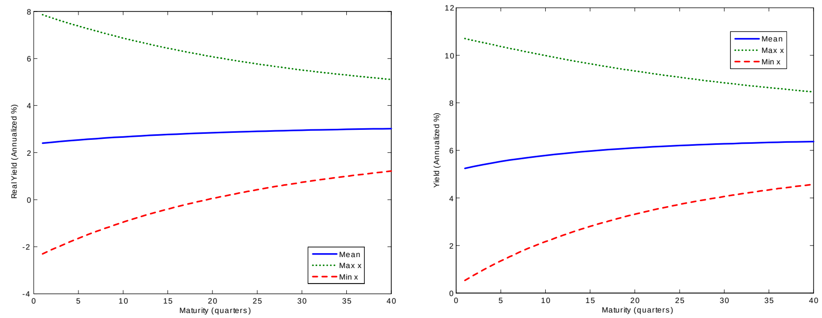

Figure 4: Responses of yield curves to x_t. The left hand figure shows the response of the real yield curve, and the right hand figure shows the response of the nominal yield curve, as x_t is varied between its sample minimum and maximum while all other state variables are held fixed at their sample means.

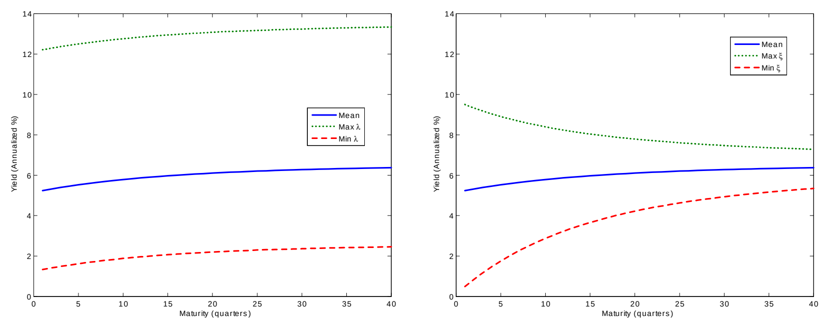

Figure 5: Responses of yield curves to λ_t and ξ_t. The left hand figure shows the response of the nominal yield curve to the permanent component of expected inflation λ_t, and the right hand figure shows the response to the transitory component of expected inflation ξ_t, as these state variables are varied between their sample minima and maxima while all other state variables are held fixed at their sample means.

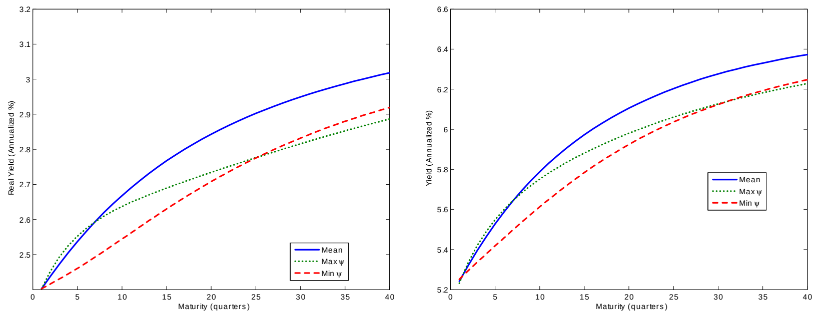

Figure 6: Responses of yield curves to ψ_t . The left hand figure shows the response of the real yield curve, and the right hand figure shows the response of the nominal yield curve, to ψ_t as it is varied between its sample minima and maxima while all other state variables are held fixed at their sample means.

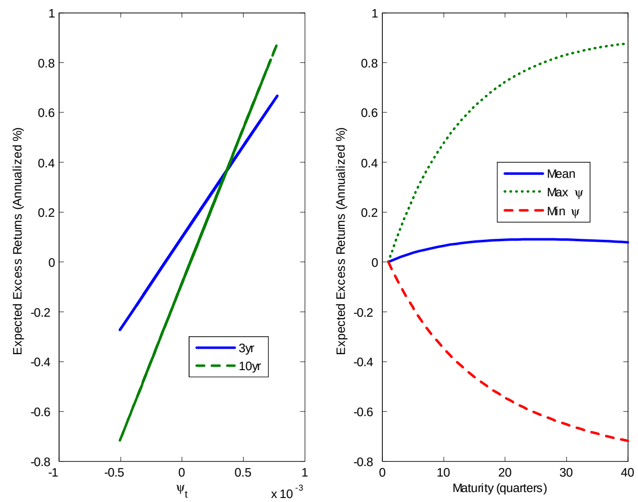

Figure 7: Responses of nominal expected excess returns to ψ_t. The left hand figure shows the expected excess returns on 3-year and 10-year nominal bonds over 3-month Treasury bills, as functions of ψ_t. The right hand figure shows the term structure of expected excess nominal bond returns as ψ_t is varied between its sample minimum and maximum while all other state variables are held fixed at their sample means.

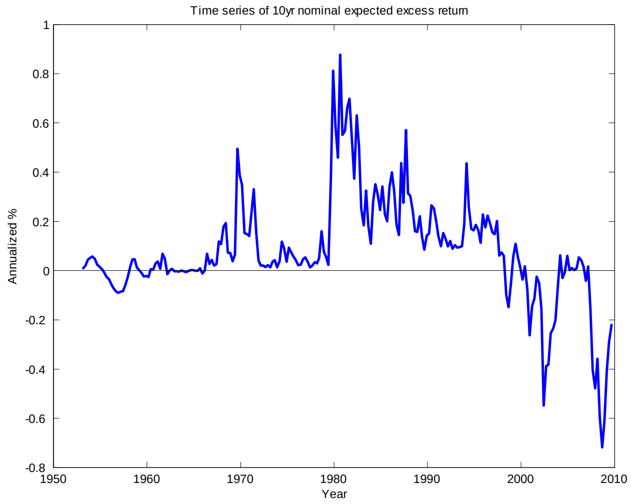

Figure 8: Time series of expected excess returns for 10-year nominal bonds.

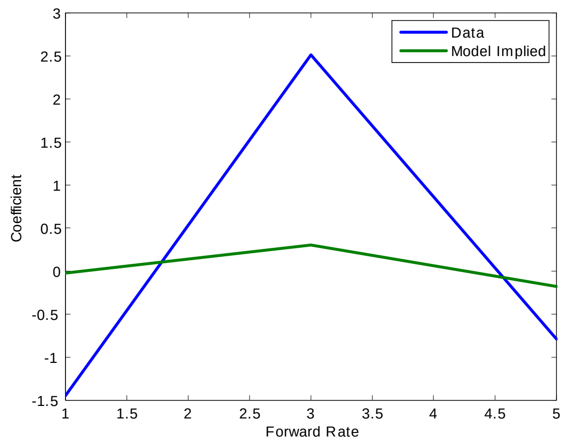

Figure 9: Data- and model- implied Cochrane-Piazzesi relationships.

Table 1: Parameter estimates.

Parameter Estimates		
Parameter	Estimate	Std Err
μ_x x 10^3	8.724	0.415
μ_ψ x 10^3	3.074	0.733
ϕ_x	0.940	0.006
ϕ_ξ	0.885	0.008
ϕ_ψ	0.793	0.032
σ_m x 10^2	4.869	3.883
σ_X x 10^3	1.124	0.104
σ_x x 10^1	4.659	0.587
σ_λ x 10^4	4.783	0.675
σ_Λ x 10^4	4.279	1.542
σ_ξ x 10^1	3.374	0.740
σ_ψ x 10^3	2.299	0.207
β_{eX}	4.303	9.745
β_{ex} x 10^2	-0.002	4.229
β_{em} x 10^2	8.855	6.963
$\rho_{x\xi}$	0.000	0.113
ρ_{xm}	-0.331	0.260
ρ_{Xm} x 10^2	-0.022	0.028
$\rho_{x\pi}$	0.001	0.458
$\rho_{\lambda m}$	-0.717	0.345
$\rho_{\Lambda m}$	0.263	0.227
$\rho_{\xi m}$	-0.259	0.222
$\rho_{\xi\pi}$	0.007	0.871
$\rho_{m\pi}$	0.037	0.144

Table 2: Sample and Implied Moments. Yield spreads (YS) are calculated over the 3mo yield. Realized excess returns (RXR) are calculated over a 3mo holding period, in excess of the 1yr yield. Units are annualized percentage points. Simulation columns report means across 1000 replications, each of which simulates a time-series of 250 quarters. The $\sigma(\widehat{CP})$ row reports the standard deviation of the fitted values from a Cochrane-Piazzesi style regression of RXR on the 1-, 3-, and 5-yr forward rates at the beginning of the holding period. The $\sigma(\widehat{CS})$ row reports the standard deviation of the fitted values from a Campbell-Shiller style regression of RXR on the same-maturity YS at the beginning of the holding period. In the rightmost column we report the fraction of simulation runs where the simulated value exceeds the data value. † Data moments for the 10yr return require 117mo yields. We interpolate the 117mo yield linearly between the 5yr and the 10yr ‡ TIPS entries refer to the 10yr spliced TIPS yield. We have this data 1/1999-9/2009.

Sample and Implied Moments

Moment	Actual Data	Model	Above
3yr YS mean	0.62	0.46	0.29
10yr YS mean	1.15	0.69	0.24
3yr YS stdev	0.45	0.46	0.52
10yr YS stdev	0.70	0.97	0.94
3yr RXR mean	1.17	0.91	0.32
10yr RXR mean	2.21	1.30	0.21
3yr RXR stdev	4.37	4.74	0.77
10yr RXR stdev	11.16	8.86	0.01
10yr TIPS yield mean	2.58‡	3.42	0.98
10yr TIPS YS mean		-0.05	
10yr TIPS RXR mean		0.16	
10yr TIPS RXR stdev		7.63	

Predictive Regressions

Moment	Actual Data	Model	Above
3yr EXR stdev		0.07	
10yr EXR stdev		0.11	
10yr TIPS EXR stdev		0.08	
3yr RXR $\sigma(\widehat{CS})$	1.04	0.30	0.00
10yr RXR $\sigma(\widehat{CS})$	2.51†	0.52	0.00
10yr TIPS RXR $\sigma(\widehat{CS})$		0.47	
3yr RXR $\sigma(\widehat{CP})$	0.79	0.60	0.23
10yr RXR $\sigma(\widehat{CP})$	2.06†	1.12	0.04

Table 3: Sample and Implied Moments. Yield spreads (YS) are calculated over the 1yr yield. Realized excess returns (RXR) are calculated over a 1yr holding period, in excess of the 1yr yield. Units are annualized percentage points. Simulation columns report means across 1000 replications, each of which simulates a time-series of 250 quarters. The $\sigma(\widehat{CP})$ row reports the standard deviation of the fitted values from a Cochrane-Piazzesi style regression of RXR on the 1-, 3-, and 5-yr forward rates at the beginning of the holding period. The $\sigma(\widehat{CS})$ row reports the standard deviation of the fitted values from a Campbell-Shiller style regression of RXR on the same-maturity YS at the beginning of the holding period. In the rightmost column we report the fraction of simulation runs where the simulated value exceeds the data value. † Data moments for the 10yr return require 9yr yields. We interpolate the 9yr yield linearly between the 5yr and the 10yr. ‡ TIPS entries refer to the 10yr spliced TIPS yield. We have this data 1/1999-9/2009.

Sample and Implied Moments

Moment	Actual Data	Model	Above
3yr YS mean	0.37	0.27	0.31
10yr YS mean	0.90	0.47	0.24
3yr YS stdev	0.51	0.63	0.85
10yr YS stdev	1.11	1.67	0.97
3yr RXR mean	0.75	0.56	0.33
10yr RXR mean	1.84†	0.96	0.21
3yr RXR stdev	3.17	3.40	0.67
10yr RXR stdev	10.32†	8.03	0.02
10yr TIPS yield mean	2.58‡	3.45	0.98
10yr TIPS YS mean		-0.11	
10yr TIPS RXR mean		0.00	
10yr TIPS RXR stdev		6.81	

Predictive Regressions

Moment	Actual Data	Model	Above
3yr EXR stdev		0.06	
10yr EXR stdev		0.16	
10yr TIPS EXR stdev		0.14	
3yr RXR $\sigma(\widehat{CS})$	0.91	0.36	0.03
10yr RXR $\sigma(\widehat{CS})$	3.54†	0.81	0.00
10yr TIPS RXR $\sigma(\widehat{CS})$		0.74	
3yr RXR $\sigma(\widehat{CP})$	1.16	0.77	0.12
10yr RXR $\sigma(\widehat{CP})$	4.51†	1.87	0.00

Table 4: Forecasting excess returns. The table below reports the R^2 for regressions in our data of actual data RXR on linear combinations of the actual data 1-, 3-, and 5-yr forward rates at the beginning of the holding period. The unconstrained column estimates the best combination in the data, and thus corresponds to the first stage of the Cochrane-Piazzesi procedure. In the other columns, the combination is restricted to be the one estimated in long-sample simulation regressions of simulated RXR on simulated forward rates. In the first panel, we allow this simulation-generated combination to be scaled up. In the second panel, we do not allow scaling. Realized excess returns (RXR) are calculated over 3mo and 1yr holding periods. † Data moments for the 10yr return require 9yr yields. These yields are in our dataset 8/1971-9/2009. For the earlier part of the sample we interpolate the 9yr yield linearly between the 5yr and the 10yr.

Forecasting Excess Returns

Holding Period	Moment	Unconstrained	Model
3-month	3yr RXR	0.032	0.023
3-month	10yr RXR	0.031	0.016
1-yr	3yr RXR	0.132	0.093
1-yr	10yr RXR	0.181†	0.083

Forecasting Excess Returns: No scaling

Holding Period	Moment	Unconstrained	Model
3-month	3yr RXR	0.032	0.001
3-month	10yr RXR	0.022	0.000
1-yr	3yr RXR	0.127	0.008
1-yr	10yr RXR	0.097†	0.002

A.2 Derivations for the Full Model

A.2.1 State Variable Processes

The state variables in the model follow the processes:

$$\begin{aligned}
-m_{t+1} &= x_t + \frac{1}{2} z_t^2 \sigma_m^2 + z_t \varepsilon_{m,t+1} \\
x_{t+1} &= \mu_x (1 - \phi_x) + \phi_x x_t + \psi_t \varepsilon_{x,t+1} + \varepsilon_{X,t+1} \\
z_{t+1} &= \mu_z (1 - \phi_z) + \phi_z z_t + \varepsilon_{z,t+1}
\end{aligned}$$

$$\begin{aligned}
\pi_{t+1} &= \lambda_t + \xi_t + \frac{1}{2} \psi_t^2 \sigma_\pi^2 + \psi_t \varepsilon_{\pi,t+1} \\
\lambda_{t+1} &= \lambda_t + \psi_t \varepsilon_{\lambda,t+1} + \varepsilon_{\Lambda,t+1} \\
\xi_{t+1} &= \phi_\xi \xi_t + \psi_t \varepsilon_{\xi,t+1} \\
\psi_{t+1} &= \mu_\psi \left(1 - \phi_\psi\right) + \phi_\psi \psi_t + \varepsilon_{\psi,t+1}
\end{aligned}$$

A.2.2 Pricing Equations

Real Term Structure The price of a single-period zero-coupon real bond satisfies

$$P_{1,t} = E_t \left[\exp\left\{m_{t+1}\right\}\right] = -x_t - \frac{1}{2} z_t^2 \sigma_m^2 + \frac{1}{2} z_t^2 \sigma_m^2 = -x_t$$

We conjecture that the price function is exponential affine in x_t and z_t with the form

$$P_{n,t} = \exp\left\{A_n + B_{x,n} x_t + B_{z,n} z_t + B_{\psi,n} \psi_t + C_{z,n} z_t^2 + C_{\psi,n} \psi_t^2 + C_{z\psi,n} z_t \psi_t\right\}.$$

The standard pricing equation implies

$$\begin{aligned}
P_{n,t} &= E_t \left[\exp\left\{p_{n-1,t+1} + m_{t+1}\right\}\right] = E_t \left[\exp\left\{ \begin{array}{c} A_{n-1} + B_{x,n-1} x_{t+1} + B_{z,n-1} z_{t+1} + B_{\psi,n-1} \psi_{t+1} + C_{z,n-1} z_{t+1}^2 + C_{\psi,n-1} \psi_{t+1}^2 \\ + C_{z\psi,n-1} z_{t+1} \psi_{t+1} - x_t - \frac{1}{2} z_t^2 \sigma_m^2 - z_t \varepsilon_{m,t+1} \end{array} \right\}\right] \qquad (1) \\
&= \exp\left\{ \begin{array}{c} A_{n-1} + B_{x,n-1} \left(\left(1 - \phi_x\right) \mu_x + \phi_x x_t\right) + B_{z,n-1} \left(\left(1 - \phi_z\right) \mu_z + \phi_z z_t\right) + B_{\psi,n-1} \left(\left(1 - \phi_\psi\right) \mu_\psi + \phi_\psi \psi_t\right) + C_{z,n-1} \left(\mu_z \left(1 - \phi_z\right) + \phi_z z_t\right)^2 \\ + C_{\psi,n-1} \left(\mu_\psi \left(1 - \phi_\psi\right) + \phi_\psi \psi_t\right)^2 + C_{z\psi,n-1} \left(\mu_z \left(1 - \phi_z\right) + \phi_z z_t\right) \left(\mu_\psi \left(1 - \phi_\psi\right) + \phi_\psi \psi_t\right) - x_t - \frac{1}{2} z_t^2 \sigma_m^2 \end{array} \right\} \\
&\quad \times E_t \left[\exp\left\{\mathbf{d}_1' \boldsymbol{\omega}_{t+1} + \boldsymbol{\omega}_{t+1}' \mathbf{D}_2 \boldsymbol{\omega}_{t+1}\right\}\right]
\end{aligned}$$

where $\boldsymbol{\omega}'_{t+1} = (\varepsilon_{X,t+1}, \varepsilon_{m,t+1}, \varepsilon_{x,t+1}, \varepsilon_{z,t+1}, \varepsilon_{\psi,t+1}) \tilde{} N(0, \boldsymbol{\Sigma}_\omega)$,

$$\mathbf{d}_1 = \begin{pmatrix} B_{x,n-1} \\ -z_t \\ B_{x,n-1}\psi_t \\ B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) \\ B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) \end{pmatrix}$$

$$\mathbf{D}_2 = \begin{pmatrix} 0 & \cdots & & & 0 \\ \vdots & \ddots & & & \\ & & & & \\ & & & C_{z,n-1} & \frac{1}{2}C_{z\psi,n-1} \\ 0 & & & \frac{1}{2}C_{z\psi,n-1} & C_{\psi,n-1} \end{pmatrix}$$

Following Campbell, Chan, and Viceira (2003), we complete the square to calculate

$$\begin{aligned} E_t\left[\exp\left\{\mathbf{d}_1'\boldsymbol{\omega}_{t+1} + \boldsymbol{\omega}'_{t+1}\mathbf{D}_2\boldsymbol{\omega}_{t+1}\right\}\right] &= \frac{|\boldsymbol{\Sigma}_\omega|^{-1/2}}{\left|\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right|^{1/2}}\exp\left\{\frac{1}{2}\mathbf{d}_1\left(\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right)^{-1}\mathbf{d}_1'\right\} \\ &= \exp\left\{-\frac{1}{2}\log|\boldsymbol{\Sigma}_\omega| + \frac{1}{2}\log|\mathbf{G}| + \frac{1}{2}\mathbf{d}_1\mathbf{G}\mathbf{d}_1'\right\} \end{aligned}$$

where $\mathbf{G} = \left(\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right)^{-1}$. Let g_{ij} be the ij-th element of $\mathbf{G}$. Then expanding and collecting terms gives

$$p_{n,t} = \left[\begin{array}{c} A_{n-1} + B_{x,n-1}((1-\phi_x)\mu_x + \phi_x x_t) + B_{z,n-1}((1-\phi_z)\mu_z + \phi_z z_t) + B_{\psi,n-1}\left((1-\phi_\psi)\mu_\psi + \phi_\psi\psi_t\right) \\ +C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)^2 + C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)^2 + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) \\ -x_t - \frac{1}{2}z_t^2\sigma_m^2 - \frac{1}{2}\log|\boldsymbol{\Sigma}_\omega| + \frac{1}{2}\log|\mathbf{G}| + \frac{1}{2}g_{11}B_{x,n-1}^2 + \frac{1}{2}g_{22}z_t^2 + \frac{1}{2}g_{33}B_{x,n-1}^2\psi_t^2 \\ +\frac{1}{2}g_{44}\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right)^2 \\ +\frac{1}{2}g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right)^2 \\ -g_{12}B_{x,n-1}z_t + g_{13}B_{x,n-1}^2\psi_t + g_{14}B_{x,n-1}\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right) \\ +g_{15}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right) \\ -g_{23}B_{x,n-1}z_t\psi_t - g_{24}z_t\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right) \\ -g_{25}z_t\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right) \\ +g_{34}B_{x,n-1}\psi_t\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right) \\ +g_{35}B_{x,n-1}\psi_t\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right) \\ +g_{45}\left(B_{z,n-1} + 2C_{z,n-1}(\mu_z(1-\phi_z) + \phi_z z_t) + C_{z\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right)\right) \\ \times\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi(1-\phi_\psi) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}(\mu_z(1-\phi_z) + \phi_z z_t)\right) \end{array}\right]$$

Thus, equating coefficients across equation (1) yields

$$
\begin{aligned}
A_n &= \left[\begin{array}{c}
A_{n-1} + B_{x,n-1}(1-\phi_x)\mu_x + B_{z,n-1}(1-\phi_z)\mu_z + B_{\psi,n-1}\left(1-\phi_\psi\right)\mu_\psi \\
+C_{z,n-1}\mu_z^2(1-\phi_z)^2 + C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 + C_{z\psi,n-1}\mu_z(1-\phi_z)\mu_\psi\left(1-\phi_\psi\right) \\
-\frac{1}{2}\log|\mathbf{\Sigma}_\omega| + \frac{1}{2}\log|\mathbf{G}| + \frac{1}{2}g_{11}B_{x,n-1}^2 + \frac{1}{2}g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 \\
+\frac{1}{2}g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right)^2 \\
+g_{14}B_{x,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + g_{15}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \\
+g_{45}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right)
\end{array}\right] \\
B_{x,n} &= B_{x,n-1}\phi_x - 1 \\
B_{z,n} &= \left[\begin{array}{c}
B_{z,n-1}\phi_z + 2C_{z,n-1}\mu_z(1-\phi_z)\phi_z + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z + 2g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z,n-1}\phi_z \\
+g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right)C_{z\psi,n-1}\phi_z - g_{12}B_{x,n-1} + 2g_{14}B_{x,n-1}C_{z,n-1}\phi_z + g_{15}B_{x,n-1}C_{z\psi,n-1}\phi_z \\
-g_{24}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) - g_{25}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \\
+g_{45}\left[\begin{array}{c}
2C_{z,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \\
+C_{z\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)
\end{array}\right]\phi_z
\end{array}\right] \\
B_{\psi,n} &= \left[\begin{array}{c}
B_{\psi,n-1}\phi_\psi + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + C_{z\psi,n-1}\mu_z(1-\phi_z)\phi_\psi + g_{13}B_{x,n-1}^2 + g_{14}B_{x,n-1}C_{z\psi,n-1}\phi_\psi + 2g_{15}B_{x,n-1}C_{\psi,n-1}\phi_\psi \\
+g_{44}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z\psi,n-1}\phi_\psi \\
+2g_{55}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right)C_{\psi,n-1}\phi_\psi \\
+g_{34}B_{x,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + g_{35}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right) \\
+g_{45}\left[\begin{array}{c}
2C_{\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z(1-\phi_z) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+C_{z\psi,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z(1-\phi_z)\right)
\end{array}\right]\phi_\psi
\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
C_{z,n} &= \left[C_{z,n-1}\phi_z^2 - \frac{1}{2}\sigma_m^2 + \frac{1}{2}g_{22} + 2g_{44}C_{z,n-1}^2\phi_z^2 + \frac{1}{2}g_{55}C_{z\psi,n-1}^2\phi_z^2 - 2g_{24}C_{z,n-1}\phi_z - g_{25}C_{z\psi,n-1}\phi_z + 2g_{45}C_{z,n-1}C_{z\psi,n-1}\phi_z^2\right] \\
C_{\psi,n} &= \left[C_{\psi,n-1}\phi_\psi^2 + \frac{1}{2}g_{33}B_{x,n-1}^2 + \frac{1}{2}g_{44}C_{z\psi,n-1}^2\phi_\psi^2 + 2g_{55}C_{\psi,n-1}^2\phi_\psi^2 + g_{34}B_{x,n-1}C_{z\psi,n-1}\phi_\psi + 2g_{35}B_{x,n-1}C_{\psi,n-1}\phi_\psi + 2g_{45}C_{\psi,n-1}C_{z\psi,n-1}\phi_\psi^2\right] \\
C_{z\psi,n} &= \left[\begin{array}{c}
C_{z\psi,n-1}\phi_z\phi_\psi + 2g_{44}C_{z,n-1}C_{z\psi,n-1}\phi_z\phi_\psi + 2g_{55}C_{\psi,n-1}C_{z\psi,n-1}\phi_z\phi_\psi - g_{23}B_{x,n-1} - g_{24}C_{z\psi,n-1}\phi_\psi - 2g_{25}C_{\psi,n-1}\phi_\psi \\
+2g_{34}B_{x,n-1}C_{z,n-1}\phi_z + g_{35}B_{x,n-1}C_{z\psi,n-1}\phi_z + g_{45}C_{z\psi,n-1}^2\phi_\psi\phi_z
\end{array}\right]
\end{aligned}
$$

Nominal Term Structure The price of a single-period zero-coupon nominal bond satisfies

$$
P_{1,t}^{\$} = E_t\left[\exp\left\{m_{t+1} - \pi_{t+1}\right\}\right] = \exp\left\{-x_t - \lambda_t - \xi_t + z_t\psi_t\sigma_{m\pi}\right\}
$$

since $z_t\varepsilon_{m,t+1}$ and $\psi_t\varepsilon_{\pi,t+1}$ are jointly conditional normal.

We now guess that the price function is exponential linear-quadratic in the state variables with the following form:

$$
P_{n,t}^{\$} = \exp\left\{A_n^{\$} + B_{x,n}^{\$}x_t + B_{z,n}^{\$}z_t + B_{\lambda,n}^{\$}\lambda_t + B_{\xi,n}^{\$}\xi_t + B_{\psi,n}^{\$}\psi_t + C_{z,n}^{\$}z_t^2 + C_{\psi,n}^{\$}\psi_t^2 + C_{z\psi,n}^{\$}z_t\psi_t\right\}
$$

The standard pricing equation then implies

$$
\begin{aligned}
P_{n,t}^{\$} &= E_t\left[\exp\left\{p_{n-1,t+1}^{\$}+m_{t+1}-\pi_{t+1}\right\}\right] \qquad (2)\\
&= E_t\left[\exp\left\{\begin{array}{c} A_{n-1}^{\$}+B_{x,n-1}^{\$}x_{t+1}+B_{z,n-1}^{\$}z_{t+1}+B_{\lambda,n-1}^{\$}\lambda_{t+1}+B_{\xi,n-1}^{\$}\xi_{t+1}+B_{\psi,n-1}^{\$}\psi_{t+1} \\ +C_{z,n-1}^{\$}z_{t+1}^2+C_{\psi,n-1}^{\$}\psi_{t+1}^2+C_{z\psi,n-1}^{\$}z_{t+1}\psi_{t+1} \\ -x_t-\frac{1}{2}z_t^2\sigma_m^2-z_t\varepsilon_{m,t+1}-\lambda_t-\xi_t-\frac{1}{2}\psi_t^2\sigma_\pi^2-\psi_t\varepsilon_{\pi,t+1}\end{array}\right\}\right]\\
&= \exp\left\{\begin{array}{c} A_{n-1}^{\$}+B_{x,n-1}^{\$}\left(\mu_x\left(1-\phi_x\right)+\phi_x x_t\right)+B_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+B_{\lambda,n-1}^{\$}\left(\mu_\lambda+\lambda_t\right)+B_{\xi,n-1}^{\$}\phi_\xi\xi_t+B_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) \\ +C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)^2+C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)^2+C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) \\ -x_t-\frac{1}{2}z_t^2\sigma_m^2-\lambda_t-\xi_t-\frac{1}{2}\psi_t^2\sigma_\pi^2\end{array}\right\}\\
&\quad\times E_t\left[\exp\left\{\mathbf{d}_1^{\$\prime}\boldsymbol{\omega}_{t+1}^{\$}+\boldsymbol{\omega}_{t+1}^{\$\prime}\mathbf{D}_2^{\$}\boldsymbol{\omega}_{t+1}^{\$}\right\}\right]
\end{aligned}
$$

where $\boldsymbol{\omega}_{t+1}^{\$\prime}=\left(\varepsilon_{X,t+1},\varepsilon_{\Lambda,t+1},\varepsilon_{\lambda,t+1},\varepsilon_{m,t+1},\varepsilon_{\pi,t+1},\varepsilon_{x,t+1},\varepsilon_{\xi,t+1},\varepsilon_{z,t+1},\varepsilon_{\psi,t+1}\right)\tilde{}N\left(0,\boldsymbol{\Sigma}_\omega^{\$}\right)$,

$$
\mathbf{d}_1^{\$}=\begin{pmatrix} B_{x,n-1}^{\$} \\ B_{\lambda,n-1}^{\$} \\ B_{\lambda,n-1}^{\$}\psi_t \\ -z_t \\ -\psi_t \\ B_{x,n-1}^{\$}\psi_t \\ B_{\xi,n-1}^{\$}\psi_t \\ B_{z,n-1}^{\$}+2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) \\ B_{\psi,n-1}^{\$}+2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\end{pmatrix}
$$

$$
\mathbf{D}_2^{\$}=\begin{pmatrix} 0 & \cdots & & 0 \\ & & & \vdots \\ \vdots & \ddots & & \\ & & C_{z,n-1}^{\$} & \frac{1}{2}C_{z\psi,n-1}^{\$} \\ 0 & \cdots & \frac{1}{2}C_{z\psi,n-1}^{\$} & C_{\psi,n-1}^{\$}\end{pmatrix}
$$

Following Campbell, Chan, and Viceira (2003), we complete the square to calculate

$$
E_t\left[\exp\left\{\mathbf{d}_1^{\$\prime}\boldsymbol{\omega}_{t+1}^{\$}+\boldsymbol{\omega}_{t+1}^{\$\prime}\mathbf{D}_2^{\$}\boldsymbol{\omega}_{t+1}^{\$}\right\}\right]=\exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}_\omega^{\$}\right|+\frac{1}{2}\log\left|\mathbf{G}^{\$}\right|+\frac{1}{2}\mathbf{d}_1^{\$}\mathbf{G}^{\$}\mathbf{d}_1^{\$\prime}\right\}
$$

where $\mathbf{G}^{\$}=\left(\boldsymbol{\Sigma}_{\omega}^{\$-1}-2\mathbf{D}_{2}^{\$}\right)^{-1}$. Let $g_{ij}^{\$}$ be the ij-th element of $\mathbf{G}$. Then expanding and collecting terms gives $g^{\$}$

$$p_{n,t}^{\$}=\left[\begin{array}{c}
A^{\$}_{n-1}+B^{\$}_{x,n-1}\left(\mu_x(1-\phi_x)+\phi_x x_t\right)+B^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+B^{\$}_{\lambda,n-1}\lambda_t+B^{\$}_{\xi,n-1}\phi_\xi\xi_t+B^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\\
+C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)^2+C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)^2+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\\
-x_t-\frac{1}{2}z_t^2\sigma_m^2-\lambda_t-\xi_t-\frac{1}{2}\psi_t^2\sigma_\pi^2-\frac{1}{2}\log\left|\boldsymbol{\Sigma}_{\omega}^{\$}\right|+\frac{1}{2}\log\left|\mathbf{G}^{\$}\right|+\frac{1}{2}g^{\$}_{11}B^{\$2}_{x,n-1}+\frac{1}{2}g^{\$}_{22}B^{\$2}_{\lambda,n-1}+\frac{1}{2}g^{\$}_{33}B^{\$2}_{\lambda,n-1}\psi_t^2+\frac{1}{2}g^{\$}_{44}z_t^2\\
+\frac{1}{2}g^{\$}_{55}\psi_t^2+\frac{1}{2}g^{\$}_{66}B^{\$2}_{x,n-1}\psi_t^2+\frac{1}{2}g^{\$}_{77}B^{\$2}_{\xi,n-1}\psi_t^2+\frac{1}{2}g^{\$}_{88}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)^2\\
+\frac{1}{2}g^{\$}_{99}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)^2+g_{12}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}+g_{13}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\psi_t\\
-g_{14}B^{\$}_{x,n-1}z_t-g_{15}B^{\$}_{x,n-1}\psi_t+g_{16}B^{\$2}_{x,n-1}\psi_t+g_{17}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\psi_t\\
+g_{18}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+g_{19}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)+g^{\$}_{23}B^{\$2}_{\lambda,n-1}\psi_t-g^{\$}_{24}B^{\$}_{\lambda,n-1}z_t-g^{\$}_{25}B^{\$}_{\lambda,n-1}\psi_t\\
+g^{\$}_{26}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1}\psi_t+g^{\$}_{27}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\psi_t+g^{\$}_{28}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+g^{\$}_{29}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)-g^{\$}_{34}B^{\$}_{\lambda,n-1}z_t\psi_t-g^{\$}_{35}B^{\$}_{\lambda,n-1}\psi_t^2\\
+g^{\$}_{36}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1}\psi_t^2+g^{\$}_{37}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\psi_t^2+g^{\$}_{38}B^{\$}_{\lambda,n-1}\psi_t\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+g^{\$}_{39}B^{\$}_{\lambda,n-1}\psi_t\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)+g^{\$}_{45}z_t\psi_t-g^{\$}_{46}B^{\$}_{x,n-1}z_t\psi_t\\
-g^{\$}_{47}B^{\$}_{\xi,n-1}z_t\psi_t-g^{\$}_{48}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)z_t\\
-g^{\$}_{49}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)z_t-g^{\$}_{56}B^{\$}_{x,n-1}\psi_t^2-g^{\$}_{57}B^{\$}_{\xi,n-1}\psi_t^2\\
-g^{\$}_{58}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\psi_t\\
-g^{\$}_{59}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)\psi_t+g^{\$}_{67}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\psi_t^2\\
+g^{\$}_{68}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)B^{\$}_{x,n-1}\psi_t\\
+g^{\$}_{69}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)B^{\$}_{x,n-1}\psi_t\\
+g^{\$}_{78}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)B^{\$}_{\xi,n-1}\psi_t\\
+g^{\$}_{79}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)B^{\$}_{\xi,n-1}\psi_t\\
+g^{\$}_{89}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
\times\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z(1-\phi_z)+\phi_z z_t\right)\right)
\end{array}\right]$$

Thus, the coefficients of the pricing equation satisfy

$$A_n^{\$} = \left[\begin{array}{c} A_{n-1}^{\$} + B_{x,n-1}^{\$}\mu_x(1-\phi_x) + B_{z,n-1}^{\$}\mu_z(1-\phi_z) + B_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z,n-1}^{\$}\mu_z^2(1-\phi_z)^2 + C_{\psi,n-1}^{\$}\mu_\psi^2(1-\phi_\psi)^2 + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\mu_\psi(1-\phi_\psi) \\ -\frac{1}{2}\log\left|\mathbf{\Sigma}_\omega^{\$}\right| + \frac{1}{2}\log\left|\mathbf{G}^{\$}\right| + \frac{1}{2}g_{11}^{\$}B_{x,n-1}^{\$2} + \frac{1}{2}g_{22}^{\$}B_{\lambda,n-1}^{\$2} + g_{12}B_{x,n-1}^{\$}B_{\lambda,n-1}^{\$} + g_{18}B_{x,n-1}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right) \\ +g_{19}B_{x,n-1}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right) + \frac{1}{2}g_{88}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)^2 \\ +\frac{1}{2}g_{99}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right)^2 + g_{28}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)B_{\lambda,n-1}^{\$} \\ +g_{29}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right)B_{\lambda,n-1}^{\$} \\ +g_{89}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right) \end{array} \right]$$

$$B_{x,n}^{\$} = B_{x,n-1}^{\$}\phi_x - 1$$

$$B_{\lambda,n}^{\$} = B_{\lambda,n-1}^{\$} - 1$$

$$B_{\xi,n}^{\$} = B_{\xi,n-1}^{\$}\phi_\xi - 1$$

$$B_{z,n}^{\$} = \left[\begin{array}{c} \left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)\phi_z - g_{14}B_{x,n-1}^{\$} + 2g_{18}B_{x,n-1}^{\$}C_{z,n-1}^{\$}\phi_z + g_{19}B_{x,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z - g_{24}^{\$}B_{\lambda,n-1}^{\$} \\ +2g_{88}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)C_{z,n-1}^{\$}\phi_z + g_{99}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right)C_{z\psi,n-1}^{\$}\phi_z \\ +2g_{28}^{\$}B_{\lambda,n-1}^{\$}C_{z,n-1}^{\$}\phi_z + g_{29}^{\$}B_{\lambda,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_z - g_{48}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right) \\ -g_{49}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right) \\ +g_{89}^{\$}\left(2C_{z,n-1}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right) + C_{z\psi,n-1}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)\right)\phi_z \end{array} \right]$$

$$B_{\psi,n}^{\$} = \left[\begin{array}{c} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right)\phi_\psi + g_{13}B_{x,n-1}^{\$}B_{\lambda,n-1}^{\$} - g_{15}B_{x,n-1}^{\$} + g_{16}B_{x,n-1}^{\$2} + g_{17}B_{x,n-1}^{\$}B_{\xi,n-1}^{\$} + g_{18}B_{x,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_\psi \\ +2g_{19}B_{x,n-1}^{\$}C_{\psi,n-1}^{\$}\phi_\psi g_{23}^{\$}B_{\lambda,n-1}^{\$2} - g_{25}^{\$}B_{\lambda,n-1}^{\$} + g_{26}^{\$}B_{\lambda,n-1}^{\$}B_{x,n-1}^{\$} + g_{27}^{\$}B_{\lambda,n-1}^{\$}B_{\xi,n-1}^{\$} + g_{28}^{\$}B_{\lambda,n-1}^{\$}C_{z\psi,n-1}^{\$}\phi_\psi + 2g_{29}^{\$}B_{\lambda,n-1}^{\$}C_{\psi,n-1}^{\$}\phi_\psi \\ +g_{38}^{\$}B_{\lambda,n-1}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right) + g_{39}^{\$}B_{\lambda,n-1}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right) \\ +g_{88}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)C_{z\psi,n-1}^{\$}\phi_\psi + 2g_{99}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right)C_{\psi,n-1}^{\$}\phi_\psi \\ -g_{58}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right) - g_{59}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right) \\ +g_{68}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)B_{x,n-1}^{\$} + g_{69}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right)B_{x,n-1}^{\$} \\ +g_{78}^{\$}\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)B_{\xi,n-1}^{\$} + g_{79}^{\$}\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right)B_{\xi,n-1}^{\$} \\ +g_{89}^{\$}\left(2\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\mu_z(1-\phi_z) + C_{z\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi)\right)C_{\psi,n-1}^{\$} + \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi(1-\phi_\psi) + C_{z\psi,n-1}^{\$}\mu_z(1-\phi_z)\right)C_{z\psi,n-1}^{\$}\right)\phi_\psi \end{array} \right]$$

$$C^{\$}_{z,n} = \left[C^{\$}_{z,n-1}\phi_z^2 - \frac{1}{2}\sigma_m^2 + \frac{1}{2}g^{\$}_{44} + 2g^{\$}_{88}C^{\$2}_{z,n-1}\phi_z^2 + \frac{1}{2}g^{\$}_{99}C^{\$2}_{z\psi,n-1}\phi_z^2 - 2g^{\$}_{48}C^{\$}_{z,n-1}\phi_z - g^{\$}_{49}C^{\$}_{z\psi,n-1}\phi_z + 2g^{\$}_{89}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_z^2\right]$$

$$C^{\$}_{\psi,n} = \left[\begin{array}{c} \frac{1}{2}g^{\$}_{66}B^{\$2}_{x,n-1} + g^{\$}_{36}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1} + 2g^{\$}_{39}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_\psi + g^{\$}_{38}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + g^{\$}_{37}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1} - g^{\$}_{35}B^{\$}_{\lambda,n-1} \\ +\frac{1}{2}g^{\$}_{33}B^{\$2}_{\lambda,n-1} + C^{\$}_{\psi,n-1}\phi_\psi^2 - \frac{1}{2}\sigma_\pi^2 + \frac{1}{2}g^{\$}_{55} + \frac{1}{2}g^{\$}_{77}B^{\$2}_{\xi,n-1} + \frac{1}{2}g^{\$}_{88}C^{\$2}_{z\psi,n-1}\phi_\psi^2 + 2g^{\$}_{99}C^{\$2}_{\psi,n-1}\phi_\psi^2 - g^{\$}_{56}B^{\$}_{x,n-1} - g^{\$}_{57}B^{\$}_{\xi,n-1} \\ -g^{\$}_{58}C^{\$}_{z\psi,n-1}\phi_\psi - 2g^{\$}_{59}C^{\$}_{\psi,n-1}\phi_\psi + g^{\$}_{67}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} + g^{\$}_{68}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2g^{\$}_{69}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_\psi \\ +g^{\$}_{78}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2g^{\$}_{79}B^{\$}_{\xi,n-1}C^{\$}_{\psi,n-1}\phi_\psi + 2g^{\$}_{89}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi_\psi^2 \end{array}\right]$$

$$C^{\$}_{z\psi,n} = \left[\begin{array}{c} g^{\$}_{39}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_z + 2g^{\$}_{38}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_z - g^{\$}_{34}B^{\$}_{\lambda,n-1} + C^{\$}_{z\psi,n-1}\phi_z\phi_\psi + g^{\$}_{45} - g^{\$}_{46}B^{\$}_{x,n-1} - g^{\$}_{47}B^{\$}_{\xi,n-1} + 2g^{\$}_{88}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_z\phi_\psi \\ +2g^{\$}_{99}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi_z\phi_\psi - g^{\$}_{48}C^{\$}_{z\psi,n-1}\phi_\psi - 2g^{\$}_{49}C^{\$}_{\psi,n-1}\phi_\psi - 2g^{\$}_{58}C^{\$}_{z,n-1}\phi_z - g^{\$}_{59}C^{\$}_{z\psi,n-1}\phi_z \\ +2g^{\$}_{68}B^{\$}_{x,n-1}C^{\$}_{z,n-1}\phi_z + g^{\$}_{69}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_z + 2g^{\$}_{78}B^{\$}_{\xi,n-1}C^{\$}_{z,n-1}\phi_z + g^{\$}_{79}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_z + g^{\$}_{89}\left(4C^{\$}_{z,n-1}C^{\$}_{\psi,n-1} + C^{\$2}_{z\psi,n-1}\right)\phi_z\phi_\psi \end{array}\right]$$

where $B^{\$}_{x,1} = -1, B^{\$}_{\lambda,1} = -1,\ B^{\$}_{\xi,1} = -1,\ C^{\$}_{z\psi,1} = \sigma_{m\pi}$ and all other coefficients are zero at $n = 1$.

A.2.3 Expected Excess Returns

Real Bond Premia The log expected gross excess return on an n-period zero-coupon real bond is

$$\log E_t\left[\frac{P_{n-1,t+1}}{P_{n,t}}\right]-E_t\left[r_{1,t+1}\right] = \log E_t\left[\exp\left\{p_{n-1.t+1}-p_{n,t}\right\}\right]-x_t$$

$$= \left[\begin{array}{c} A_{n-1}-A_n+B_{x,n-1}\mu_x\left(1-\phi_x\right)+B_{z,n-1}\mu_z\left(1-\phi_z\right)+B_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) \\ +C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2+C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) \\ +\left(B_{x,n-1}\phi_x-B_{x,n}-1\right)x_t+\left(C_{z,n-1}\phi_z^2-C_{z,n}\right)z_t^2+\left(C_{\psi,n-1}\phi_\psi^2-C_{\psi,n}\right)\psi_t^2+\left(C_{z\psi,n-1}\phi_z\phi_\psi-C_{z\psi,n}\right)z_t\psi_t \\ +\left(B_{z,n-1}\phi_z-B_{z,n}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t \\ +\left(B_{\psi,n-1}\phi_\psi-B_{\psi,n}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t \end{array}\right]$$

$$+\log E_t\left[\exp\left\{\begin{array}{c} B_{x,n-1}\psi_t\varepsilon_{x,t+1}+B_{x,n-1}\varepsilon_{X,t+1}+C_{z,n-1}\varepsilon_{z,t+1}^2+C_{\psi,n-1}\varepsilon_{\psi,t+1}^2+C_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right\}\right]$$

since the shocks are conditionally jointly normal. Note that the coefficient recursion implies that $B_{x,n}=B_{x,n-1}\phi_x-1$ so that the terms involving x_t drop out. Following Campbell, Chan, and Viceira (2003), we calculate the expectation by completing the square. Let $\boldsymbol{\nu}'=\left(\varepsilon_{X,t+1},\varepsilon_{x,t+1},\varepsilon_{z,t+1},\varepsilon_{\psi,t+1}\right)\tilde{}N\left(0,\boldsymbol{\Sigma}_v\right)$,

$$\mathbf{f}_1=\begin{pmatrix} B_{x,n-1} \\ B_{x,n-1}\psi_t \\ \left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\ \left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \end{pmatrix}$$

$$\mathbf{F}_2=\begin{pmatrix} 0 & 0 & 0 \\ & & \\ 0 & C_{z,n-1} & \frac{1}{2}C_{z\psi,n-1} \\ 0 & \frac{1}{2}C_{z\psi,n-1} & C_{\psi,n-1} \end{pmatrix}$$

Then

$$E_t\left[\exp\left\{\mathbf{f}_1'\boldsymbol{\nu}+\boldsymbol{\nu}'\mathbf{F}_2\boldsymbol{\nu}\right\}\right]=\exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}_\nu\right|+\frac{1}{2}\log\left|\mathbf{H}\right|+\frac{1}{2}\mathbf{f}_1\mathbf{H}\mathbf{f}_1'\right\}$$

where $\mathbf{H}=\left(\boldsymbol{\Sigma}_\nu^{-1}-2\mathbf{F}_2\right)^{-1}$.

Let h_{ij} be the ij-th element of $\mathbf{H}$. Then expanding and collecting terms gives

$$\log E_t\left[\frac{P_{n-1,t+1}}{P_{n,t}}\right]-E_t\left[r_{1,t+1}\right]=\left[\begin{array}{c}
A_{n-1}-A_n+B_{x,n-1}\mu_x\left(1-\phi_x\right)+B_{z,n-1}\mu_z\left(1-\phi_z\right)+B_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\\
+C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2+C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right)\\
+\left(C_{z,n-1}\phi_z^2-C_{z,n}\right)z_t^2+\left(C_{\psi,n-1}\phi_\psi^2-C_{\psi,n}\right)\psi_t^2+\left(C_{z\psi,n-1}\phi_z\phi_\psi-C_{z\psi,n}\right)z_t\psi_t\\
+\left(B_{z,n-1}\phi_z-B_{z,n}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t\\
+\left(B_{\psi,n-1}\phi_\psi-B_{\psi,n}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t\\
-\frac{1}{2}\log\left|\mathbf{\Sigma}_\nu\right|+\frac{1}{2}\log\left|\mathbf{H}\right|+\frac{1}{2}h_{11}B_{x,n-1}^2+\frac{1}{2}h_{22}B_{x,n-1}^2\psi_t^2\\
+\frac{1}{2}h_{33}\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)^2\\
+\frac{1}{2}h_{44}\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)^2\\
+h_{12}B_{x,n-1}^2\psi_t+h_{13}B_{x,n-1}\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h_{14}B_{x,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\\
+h_{23}B_{x,n-1}\psi_t\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h_{24}B_{x,n-1}\psi_t\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\\
+h_{34}\left(B_{z,n-1}+2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
\times\left(B_{\psi,n-1}+2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)
\end{array}\right]$$

Thus, we can write

$$\log E_t\left[\frac{P_{n-1,t+1}}{P_{n,t}}\right]-E_t\left[r_{1,t+1}\right]=\kappa_n+\eta_{z,n}z_t+\eta_{\psi,n}\psi_t+\beta_{z,n}z_t^2+\beta_{\psi,n}\psi_t^2+\beta_{z\psi,n}z_t\psi_t$$

where the coefficients are given by

$$\kappa_n=\left[\begin{array}{c}
A_{n-1}-A_n+B_{x,n-1}\mu_x\left(1-\phi_x\right)+B_{z,n-1}\mu_z\left(1-\phi_z\right)+B_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2+C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2\\
+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right)-\frac{1}{2}\log\left|\mathbf{\Sigma}_\nu\right|+\frac{1}{2}\log\left|\mathbf{H}\right|+\frac{1}{2}h_{11}B_{x,n-1}^2+\frac{1}{2}h_{33}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2\\
+\frac{1}{2}h_{44}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2+h_{13}B_{x,n-1}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\\
+h_{14}B_{x,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\\
+h_{34}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array}\right]$$

$$\eta_{z,n}=\left[\begin{array}{c}
B_{z,n-1}\phi_z-B_{z,n}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z+2h_{33}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z,n-1}\phi_z\\
+h_{44}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{z\psi,n-1}\phi_z+2h_{13}B_{x,n-1}C_{z,n-1}\phi_z+h_{14}B_{x,n-1}C_{z\psi,n-1}\phi_z\\
+h_{34}\left[2C_{z,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)+C_{z\psi,n-1}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\right]\phi_z
\end{array}\right]$$

$$\eta_{\psi,n}=\left[\begin{array}{c}
B_{\psi,n-1}\phi_\psi-B_{\psi,n}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi+h_{33}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z\psi,n-1}\phi_\psi\\
+2h_{44}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{\psi,n-1}\phi_\psi+h_{12}B_{x,n-1}^2\psi_t+h_{13}B_{x,n-1}C_{z\psi,n-1}\phi_\psi\psi_t+2h_{14}B_{x,n-1}C_{\psi,n-1}\phi_\psi\psi_t\\
+h_{23}B_{x,n-1}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)+h_{24}B_{x,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\\
+h_{34}\left[2C_{\psi,n-1}\left(B_{z,n-1}+2C_{z,n-1}\mu_z\left(1-\phi_z\right)+C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)+C_{z\psi,n-1}\left(B_{\psi,n-1}+2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\right]\phi_\psi
\end{array}\right]$$

$$\beta_{z,n} = \left[\left(C_{z,n-1}\phi_z^2 - C_{z,n} \right) + 2h_{33}C_{z,n-1}^2\phi_z^2 + \frac{1}{2}h_{44}C_{z\psi,n-1}^2\phi_z^2 + 2h_{34}C_{z,n-1}C_{z\psi,n-1}\phi_z^2 \right]$$

$$\beta_{\psi,n} = \left[\left(C_{\psi,n-1}\phi_\psi^2 - C_{\psi,n} \right) + \frac{1}{2}h_{33}C_{z\psi,n-1}^2\phi_\psi^2 + 2h_{44}C_{\psi,n-1}^2\phi_\psi^2 + h_{23}B_{x,n-1}C_{z\psi,n-1}\phi_\psi + 2h_{24}B_{x,n-1}C_{\psi,n-1}\phi_\psi + h_{34}2C_{\psi,n-1}C_{z\psi,n-1}\phi_\psi^2 \right]$$

$$\beta_{z\psi,n} = \left[\begin{array}{c} \left(C_{z\psi,n-1}\phi_z\phi_\psi - C_{z\psi,n} \right) + 2h_{33}C_{z,n-1}C_{z\psi,n-1}\phi_z\phi_\psi + 2h_{44}C_{\psi,n-1t}C_{z\psi,n-1}\phi_z\phi_\psi + 2h_{23}B_{x,n-1}C_{z,n-1}\phi_z \\ +h_{24}B_{x,n-1}C_{z\psi,n-1}\phi_z + h_{34}C_{z\psi,n-1}^2\phi_\psi\phi_z \end{array} \right]$$

Nominal Bond Premia The log conditional expected real return on a 1-period zero-coupon nominal bond is

$$E_t\left[r_{1,t+1}^{\$} - \pi_{t+1} \right] = -\sigma_{m,\pi} z_t \psi_t$$

The log conditional expected gross excess return on an n-period zero-coupon nominal bond is

$$\begin{aligned}
\log E_t\left[\frac{P_{n-1,t+1}^{\$}}{P_{n,t}^{\$}} \right] - E_t\left[r_{1,t+1}^{\$} \right] &= \log E_t\left[\exp\left\{ p_{n-1.t+1}^{\$} - p_{n,t}^{\$} \right\} \right] - x_t - \lambda_t - \xi_t + \sigma_{m,\pi} z_t \psi_t \\
&= \left[\begin{array}{c} A_{n-1}^{\$} - A_n^{\$} + B_{x,n-1}^{\$}\mu_x\left(1-\phi_x\right) + B_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right) + B_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right) \\ +C_{z,n-1}^{\$}\mu_z^2\left(1-\phi_z\right)^2 + C_{\psi,n-1}^{\$}\mu_\psi^2\left(1-\phi_\psi\right)^2 + C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) \\ +\left(B_{x,n-1}^{\$}\phi_x - B_{x,n}^{\$} - 1\right)x_t + \left(B_{\lambda,n-1}^{\$} - B_{\lambda,n}^{\$} - 1\right)\lambda_t + \left(B_{\xi,n-1}^{\$}\phi_\xi - B_{\xi,n}^{\$} - 1\right)\xi_t \\ +\left(C_{z,n-1}^{\$}\phi_z^2 - C_{z,n}^{\$}\right)z_t^2 + \left(C_{\psi,n-1}^{\$}\phi_\psi^2 - C_{\psi,n}^{\$}\right)\psi_t^2 + \left(\sigma_{m,\pi} + C_{z\psi,n-1}^{\$}\phi_z\phi_\psi - C_{z\psi,n}^{\$}\right)z_t\psi_t \\ +\left(B_{z,n-1}^{\$}\phi_z - B_{z,n}^{\$} + 2C_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right)\phi_z + C_{z\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t \\ +\left(B_{\psi,n-1}^{\$}\phi_\psi - B_{\psi,n}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t \end{array} \right] \\
&\quad + \log E_t\left[\exp\left\{ \begin{array}{c} B_{x,n-1}^{\$}\psi_t\varepsilon_{x,t+1} + B_{x,n-1}^{\$}\varepsilon_{X,t+1} + B_{\lambda,n-1}^{\$}\psi_t\varepsilon_{\lambda,t+1} + B_{\lambda,n-1}^{\$}\varepsilon_{\Lambda,t+1} + B_{\xi,n-1}^{\$}\psi_t\varepsilon_{\xi,t+1} \\ +C_{z,n-1}^{\$}\varepsilon_{z,t+1}^2 + C_{\psi,n-1}^{\$}\varepsilon_{\psi,t+1}^2 + C_{z\psi,n-1}^{\$}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array} \right\} \right]
\end{aligned}$$

Note that the coefficient recursions imply that $B_{x,n}^{\$} = B_{x,n-1}^{\$}\phi_x - 1$, $B_{\lambda,n}^{\$} = B_{\lambda,n-1}^{\$} - 1$,and $B_{\xi,n}^{\$} = B_{\xi,n-1}^{\$}\phi_\xi - 1$, so that the terms involving x_t, λ_t, and ξ_t drop out. Following Campbell, Chan, and Viceira (2003), we calculate the expectation by completing the square. Let

$$\boldsymbol{\nu}^{\$\prime} = \left(\varepsilon_{X,t+1},\varepsilon_{\Lambda,t+1},\varepsilon_{x,t+1},\varepsilon_{\lambda,t+1},\varepsilon_{\xi,t+1},\varepsilon_{z,t+1},\varepsilon_{\psi,t+1}\right) \tilde{} N\left(0,\boldsymbol{\Sigma}_v^{\$}\right),$$

$$\mathbf{f}_1^{\$} = \begin{pmatrix} B_{x,n-1}^{\$} \\ B_{\lambda,n-1}^{\$} \\ B_{x,n-1}^{\$}\psi_t \\ B_{\lambda,n-1}^{\$}\psi_t \\ B_{\xi,n-1}^{\$}\psi_t \\ \left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right) \\ \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right) \end{pmatrix}$$

$$\mathbf{F}_2^{\$} = \begin{pmatrix} 0 & \cdots & & 0 \\ & \ddots & & \\ \vdots & & & \\ & & C_{z,n-1}^{\$} & \frac{1}{2}C_{z\psi,n-1}^{\$} \\ 0 & & \frac{1}{2}C_{z\psi,n-1}^{\$} & C_{\psi,n-1}^{\$} \end{pmatrix}$$

Then

$$E_t\left[\exp\left\{\mathbf{f}_1^{\$\prime}\boldsymbol{\nu}^{\$} + \boldsymbol{\nu}^{\$\prime}\mathbf{F}_2^{\$}\boldsymbol{\nu}^{\$}\right\}\right] = \exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}_\nu^{\$}\right| + \frac{1}{2}\log\left|\mathbf{H}^{\$}\right| + \frac{1}{2}\mathbf{f}_1^{\$}\mathbf{H}^{\$}\mathbf{f}_1^{\$\prime}\right\}$$

where $\mathbf{H}^{\$} = \left(\boldsymbol{\Sigma}_\nu^{\$-1} - 2\mathbf{F}_2^{\$}\right)^{-1}$.

Let $h_{ij}^{\$}$ be the ij-th element of $\mathbf{H}^{\$}$. Then expanding and collecting terms gives

$$
\log E_t\left[\frac{P^{\$}_{n-1,t+1}}{P^{\$}_{n,t}}\right]-E_t\left[r^{\$}_{1,t+1}\right]=\left[\begin{array}{c}
A^{\$}_{n-1}-A^{\$}_{n}+B^{\$}_{x,n-1}\mu_x\left(1-\phi_x\right)+B^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+B^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2\\
+C^{\$2}_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right)+\left(C^{\$}_{z,n-1}\phi_z^2-C^{\$}_{z,n}\right)z_t^2+\left(C^{\$}_{\psi,n-1}\phi_\psi^2-C^{\$}_{\psi,n}\right)\psi_t^2\\
+\left(\sigma_{m,\pi}+C^{\$}_{z\psi,n-1}\phi_z\phi_\psi-C^{\$}_{z\psi,n}\right)z_t\psi_t+\left(B^{\$}_{z,n-1}\phi_z-B^{\$}_{z,n}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t\\
+\left(B^{\$}_{\psi,n-1}\phi_\psi-B^{\$}_{\psi,n}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t-\frac{1}{2}\log\left|\boldsymbol{\Sigma}^{\$}_{\nu}\right|+\frac{1}{2}\log\left|\mathbf{H}^{\$}\right|+\frac{1}{2}h^{\$}_{11}B^{\$2}_{x,n-1}\\
+\frac{1}{2}h^{\$}_{22}B^{\$2}_{\lambda,n-1}+\frac{1}{2}h^{\$}_{33}B^{\$2}_{x,n-1}\psi_t^2+\frac{1}{2}h^{\$}_{44}B^{\$2}_{\lambda,n-1}\psi_t^2+\frac{1}{2}h^{\$}_{55}B^{\$2}_{\xi,n-1}\psi_t^2\\
+\frac{1}{2}h^{\$}_{66}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)^2\\
+\frac{1}{2}h^{\$}_{77}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)^2\\
+h^{\$}_{12}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}+h^{\$}_{13}B^{\$2}_{x,n-1}\psi_t+h^{\$}_{14}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\psi_t+h^{\$}_{15}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\psi_t\\
+h^{\$}_{16}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{17}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\\
+h^{\$}_{23}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1}\psi_t+h_{24}B^{\$2}_{\lambda,n-1}\psi_t+h^{\$}_{25}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\psi_t\\
+h^{\$}_{26}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{27}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)+h^{\$}_{34}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\psi_t^2\\
+h^{\$}_{35}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\psi_t^2+h^{\$}_{36}B^{\$}_{x,n-1}\psi_t\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{37}B^{\$}_{x,n-1}\psi_t\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)+h^{\$}_{45}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\psi_t^2\\
+h^{\$}_{46}B^{\$}_{\lambda,n-1}\psi_t\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{47}B^{\$}_{\lambda,n-1}\psi_t\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\\
+h^{\$}_{56}B^{\$}_{\xi,n-1}\psi_t\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
+h^{\$}_{57}B^{\$}_{\xi,n-1}\psi_t\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)\\
+h^{\$}_{67}\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)\right)\\
\times\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right)+\phi_\psi\psi_t\right)+C^{\$}_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right)+\phi_z z_t\right)\right)
\end{array}\right]
$$

Thus, we can write

$$
\log E_t\left[\frac{P^{\$}_{n-1,t+1}}{P^{\$}_{n,t}}\right]-E_t\left[r^{\$}_{1,t+1}\right]=\kappa^{\$}_n+\eta^{\$}_{z,n}z_t+\eta^{\$}_{\psi,n}\psi_t+\beta^{\$}_{z,n}z_t^2+\beta^{\$}_{\psi,n}\psi_t^2+\beta^{\$}_{z\psi,n}z_t\psi_t
$$

where the coefficients are given by

$$
\kappa^{\$}_{n} = \left[\begin{array}{c}
A^{\$}_{n-1} - A^{\$}_{n} + B^{\$}_{x,n-1}\mu_x\left(1-\phi_x\right) + B^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + B^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2 + C^{\$2}_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 \\
+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) - \frac{1}{2}\log\left|\mathbf{\Sigma}^{\$}_{\nu}\right| + \frac{1}{2}\log\left|\mathbf{H}^{\$}\right| + \frac{1}{2}h^{\$}_{11}B^{\$2}_{x,n-1} + \frac{1}{2}h^{\$}_{22}B^{\$2}_{\lambda,n-1} \\
+\frac{1}{2}h^{\$}_{66}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 + \frac{1}{2}h^{\$}_{77}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2 \\
+h^{\$}_{12}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} + h^{\$}_{16}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+h^{\$}_{17}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{26}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+h^{\$}_{27}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{67}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array} \right]
$$

$$
\eta^{\$}_{z,n} = \left[\begin{array}{c}
B^{\$}_{z,n-1}\phi_z - B^{\$}_{z,n} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z \\
+2h^{\$}_{66}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z,n-1}\phi_z \\
+h^{\$}_{77}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{z\psi,n-1}\phi_z \\
+2h^{\$}_{16}B^{\$}_{x,n-1}C^{\$}_{z,n-1}\phi_z + h^{\$}_{17}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_z 2h^{\$}_{26}B^{\$}_{\lambda,n-1}C^{\$}_{z,n-1}\phi_z + h^{\$}_{27}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_z \\
+h^{\$}_{67}\left(\begin{array}{c}
\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z\psi,n-1}\phi_z \\
+2C^{\$}_{z,n-1}\phi_z\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array} \right)
\end{array} \right]
$$

$$
\eta^{\$}_{\psi,n} = \left[\begin{array}{c}
\left(B^{\$}_{\psi,n-1}\phi_\psi - B^{\$}_{\psi,n} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right) + h^{\$}_{66}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z\psi,n-1}\phi_\psi \\
+2h^{\$}_{77}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{\psi,n-1}\phi_\psi + h^{\$}_{13}B^{\$2}_{x,n-1} + h^{\$}_{14}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} + h^{\$}_{15}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} \\
+h^{\$}_{16}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2h^{\$}_{17}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_\psi + h^{\$}_{23}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1} + h^{\$}_{24}B^{\$2}_{\lambda,n-1} + h^{\$}_{25}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1} + h^{\$}_{26}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_\psi + 2h^{\$}_{27}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_\psi \\
+h^{\$}_{36}B^{\$}_{x,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + h^{\$}_{37}B^{\$}_{x,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{46}B^{\$}_{\lambda,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + h^{\$}_{47}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{56}B^{\$}_{\xi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + h^{\$}_{57}B^{\$}_{\xi,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+h^{\$}_{67}\left[\begin{array}{c}
2C^{\$}_{\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+C^{\$}_{z\psi,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array} \right]\phi_\psi
\end{array} \right]
$$

$$
\beta^{\$}_{z,n} = \left[C^{\$}_{z,n-1}\phi_z^2 - C^{\$}_{z,n} + 2h^{\$}_{66}C^{\$2}_{z,n-1}\phi_z^2 + \frac{1}{2}h^{\$}_{77}C^{\$2}_{z\psi,n-1}\phi_z^2 + 2h^{\$}_{67}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_z^2\right]
$$

$$\beta^{\$}_{\psi,n} = \left[\begin{array}{c} C^{\$}_{\psi,n-1}\phi^2_{\psi} - C^{\$}_{\psi,n} + \frac{1}{2}h^{\$}_{33}B^{\$2}_{x,n-1} + \frac{1}{2}h^{\$}_{44}B^{\$2}_{\lambda,n-1} + \frac{1}{2}h^{\$}_{55}B^{\$2}_{\xi,n-1} + \frac{1}{2}h^{\$}_{66}C^{\$2}_{z\psi,n-1}\phi^2_{\psi} + 2h^{\$}_{77}C^{\$2}_{\psi,n-1}\phi^2_{\psi} \\ +h^{\$}_{34}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1} + h^{\$}_{35}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1} + h^{\$}_{36}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} + 2h^{\$}_{37}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} \\ +h^{\$}_{45}B^{\$}_{\xi,n-1}B^{\$}_{\lambda,n-1} + h^{\$}_{46}B^{\$}_{\lambda,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} + 2h^{\$}_{47}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} \\ +h^{\$}_{56}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_{\psi} + 2h^{\$}_{57}B^{\$}_{\xi,n-1}C^{\$}_{\psi,n-1}\phi_{\psi} + 2h^{\$}_{67}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi^2_{\psi} \end{array} \right]$$

$$\beta^{\$}_{z\psi,n} = \left[\begin{array}{c} \sigma_{m,\pi} + C^{\$}_{z\psi,n-1}\phi_z\phi_{\psi} - C^{\$}_{z\psi,n} + 2h^{\$}_{66}C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\phi_z\phi_{\psi} + 2h^{\$}_{77}C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\phi_z\phi_{\psi} + 2h^{\$}_{35}B^{\$}_{\lambda,n-1}C^{\$}_{z,n-1}\phi_z \\ +2h^{\$}_{36}B^{\$}_{x,n-1}C^{\$}_{z,n-1}\phi_z + h^{\$}_{37}B^{\$}_{x,n-1}C^{\$}_{z\psi,n-1}\phi_z + 2h^{\$}_{46}B^{\$}_{\lambda,n-1}C^{\$}_{z,n-1}\phi_z + h^{\$}_{47}B^{\$}_{\lambda,n-1}C_{z\psi,n-1}\phi_z + 2h^{\$}_{56}B^{\$}_{\xi,n-1}C^{\$}_{z,n-1}\phi_z \\ +h^{\$}_{57}B^{\$}_{\xi,n-1}C^{\$}_{z\psi,n-1}\phi_z + h^{\$}_{67}\left(4C^{\$}_{z,n-1}C^{\$}_{\psi,n-1} + C^{\$2}_{z\psi,n-1}\right)\phi_{\psi}\phi_z \end{array} \right]$$

A.2.4 Observation Equations

Stock Returns We model the unexpected stock return as

$$r_{e,t+1} - E_t r_{e,t+1} = \beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}$$

We impose that the only non-zero covariance of $\varepsilon_{X,t+1}$ is $\sigma_{X,m}$. The standard pricing equation then implies that the expected equity return satisfies

$$\begin{aligned} 1 &= E_t\left[\exp\left(r_{e,t+1} + m_{t+1}\right)\right] \\ &= \exp\left(E_t r_{e,t+1} - x_t - \frac{1}{2}z_t^2\sigma_m^2\right)\exp\left(\begin{array}{c} \frac{1}{2}\beta_{ex}^2\sigma_x^2 + \frac{1}{2}\beta_{eX}^2\sigma_X^2 + \frac{1}{2}\beta_{em}^2\sigma_m^2 + \frac{1}{2}z_t^2\sigma_m^2 \\ +\beta_{ex}\beta_{em}\sigma_{xm} - \beta_{ex}z_t\sigma_{xm} + \beta_{eX}\beta_{em}\sigma_{X,m} - \beta_{eX}z_t\sigma_{Xm} - \beta_{em}z_t\sigma_m^2 \end{array}\right) \end{aligned}$$

so that

$$r_{e,t+1} = -\frac{1}{2}\beta_{ex}^2\sigma_x^2 - \frac{1}{2}\beta_{eX}^2\sigma_X^2 - \frac{1}{2}\beta_{em}^2\sigma_m^2 - \beta_{ex}\beta_{em}\sigma_{xm} - \beta_{eX}\beta_{em}\sigma_{X,m} + x_t + \left(\beta_{ex}\sigma_{xm} + \beta_{eX}\sigma_{Xm} + \beta_{em}\sigma_m^2\right)z_t + \beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}$$

and

$$E_t\left[r_{e,t+1} - r_{1,t+1}\right] - \frac{1}{2}Var_t\left[r_{e,t+1} - r_{1,t+1}\right] = \left(\beta_{ex}\sigma_{xm} + \beta_{eX}\sigma_{Xm} + \beta_{em}\sigma_m^2\right)z_t$$

Stock-Real Bond Return Covariance As we saw above, the holding period return on an n-period real bond is

$$\begin{aligned} r_{n,t+1} &= p_{n-1,t+1} - p_{n,t} \\ &= \left[\begin{array}{c} A_{n-1} - A_n + B_{x,n-1}\mu_x\left(1-\phi_x\right) + B_{z,n-1}\mu_z\left(1-\phi_z\right) + B_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z,n-1}\mu_z^2\left(1-\phi_z\right)^2 + C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 \\ +C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) + \left(B_{x,n-1}\phi_x - B_{x,n} - 1\right)x_t + \left(C_{z,n-1}\phi_z^2 - C_{z,n}\right)z_t^2 + \left(C_{\psi,n-1}\phi_\psi^2 - C_{\psi,n}\right)\psi_t^2 \\ + \left(C_{z\psi,n-1}\phi_z\phi_\psi - C_{z\psi,n}\right)z_t\psi_t + \left(B_{z,n-1}\phi_z - B_{z,n} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right)\phi_z + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t \\ + \left(B_{\psi,n-1}\phi_\psi - B_{\psi,n} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t \end{array}\right] \\ &\quad + \left[\begin{array}{c} B_{x,n-1}\psi_t\varepsilon_{x,t+1} + B_{x,n-1}\varepsilon_{X,t+1} + C_{z,n-1}\varepsilon_{z,t+1}^2 + C_{\psi,n-1}\varepsilon_{\psi,t+1}^2 + C_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ + \left(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ + \left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right] \end{aligned}$$

We assume that the unexpected stock return is assumed to be

$$r_{e,t+1} - E_t r_{e,t+1} = \beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}$$

Since the ε's are conditionally jointly normal and mean zero we have $Cov_t\left(\varepsilon_{a,t+1}, \varepsilon_{b,t+1}^2\right) = 0$ and $Cov_t\left(\varepsilon_{a,t+1}, \varepsilon_{b,t+1}\varepsilon_{c,t+1}\right) = 0$ for all a, b, c. Furthermore, we impose that the only non-zero covariance of $\varepsilon_{X,t+1}$ is $\sigma_{X,m}$.Thus, the expression for the conditional covariance of stock returns with returns on a long-term real bond is

$$\begin{aligned} Cov_t\left(r_{e,t+1}, r_{n,t+1}\right) &= \beta_{ex}\left(\begin{array}{c} \left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{x.z} \\ +\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{x,\psi} \end{array}\right) \\ &+\beta_{eX}B_{x,n-1}\sigma_X^2 \\ &+\beta_{em}\left(\begin{array}{c} B_{x,n-1}\sigma_{Xm} + \left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{z,m} \\ +\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{\psi,m} \end{array}\right) \\ &+\left[\beta_{ex}\left(2C_{z,n-1}\sigma_{xz}\phi_z + C_{z\psi,n-1}\sigma_{x\psi}\phi_z\right) + \beta_{em}\left(2C_{z,n-1}\sigma_{zm}\phi_z + C_{z\psi,n-1}\sigma_{\psi m}\phi_z\right)\right]z_t \\ &+\left[\beta_{ex}\left(B_{x,n-1}\sigma_x^2 + C_{z\psi,n-1}\sigma_{xz}\phi_\psi + 2C_{\psi,n-1}\sigma_{x\psi}\phi_\psi\right) + \beta_{em}\left(B_{x,n-1}\sigma_{xm} + C_{z\psi,n-1}\sigma_{zm}\phi_\psi + 2C_{\psi,n-1}\sigma_{\psi m}\phi_\psi\right)\right]\psi_t \end{aligned}$$

Stock-Nominal Bond Return Covariance As we saw above, the holding period return on an n-period nominal bond is

$$\begin{aligned} r_{n,t+1}^{\$} &= p_{n-1,t+1}^{\$} - p_{n,t}^{\$} \\ &= \left[\begin{array}{c} A_{n-1}^{\$} - A_n^{\$} + B_{x,n-1}^{\$}\mu_x\left(1-\phi_x\right) + B_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right) + B_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right) + C_{z,n-1}^{\$}\mu_z^2\left(1-\phi_z\right)^2 + C_{\psi,n-1}^{\$2}\mu_\psi^2\left(1-\phi_\psi\right)^2 \\ +C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\mu_\psi\left(1-\phi_\psi\right) + \left(B_{x,n-1}^{\$}\phi_x - B_{x,n}^{\$} - 1\right)x_t + \left(B_{\xi,n-1}^{\$}\phi_\xi - B_{\xi,n}^{\$} - 1\right)\xi_t \\ +\left(C_{z,n-1}^{\$}\phi_z^2 - C_{z,n}^{\$}\right)z_t^2 + \left(C_{\psi,n-1}^{\$}\phi_\psi^2 - C_{\psi,n}^{\$}\right)\psi_t^2 + \left(\sigma_{m,\pi} + C_{z\psi,n-1}^{\$}\phi_z\phi_\psi - C_{z\psi,n}^{\$}\right)z_t\psi_t \\ +\left(B_{z,n-1}^{\$}\phi_z - B_{z,n}^{\$} + 2C_{z,n-1}^{\$}\mu_z\left(1-\phi_z\right)\phi_z + C_{z\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\phi_z\right)z_t \\ +\left(B_{\psi,n-1}^{\$}\phi_\psi - B_{\psi,n}^{\$} + 2C_{\psi,n-1}^{\$}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + C_{z\psi,n-1}^{\$}\mu_z\left(1-\phi_z\right)\phi_\psi\right)\psi_t \end{array}\right] \\ &+\left[\begin{array}{c} B_{x,n-1}^{\$}\psi_t\varepsilon_{x,t+1} + B_{x,n-1}^{\$}\varepsilon_{X,t+1} + B_{\lambda,n-1}^{\$}\psi_t\varepsilon_{\lambda,t+1} + B_{\lambda,n-1}^{\$}\varepsilon_{\Lambda,t+1} + B_{\xi,n-1}^{\$}\psi_t\varepsilon_{\xi,t+1} \\ +C_{z,n-1}^{\$}\varepsilon_{z,t+1}^2 + C_{\psi,n-1}^{\$}\varepsilon_{\psi,t+1}^2 + C_{z\psi,n-1}^{\$}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ +\left(B_{z,n-1}^{\$} + 2C_{z,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)\varepsilon_{z,t+1} \\ +\left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{z\psi,n-1}^{\$}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array}\right] \end{aligned}$$

We assume that the unexpected stock return is assumed to be

$$r_{e,t+1} - E_t r_{e,t+1} = \beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}$$

Thus, the conditional covariance with the real return on short term nominal bond is

$$Cov_t\left(r_{e,t+1}, r_{1,t+1}^{\$} - \pi_{t+1}\right) = Cov\left(\beta_{ex}\varepsilon_{x,t+1} + \beta_{eX}\varepsilon_{X,t+1} + \beta_{em}\varepsilon_{m,t+1}, -\psi_t\varepsilon_{\pi,t+1}\right) = -\psi_t\left(\beta_{ex}\sigma_{x\pi} + \beta_{em}\sigma_{m\pi}\right)$$

since we impose the condition that the only non-zero covariance of $\varepsilon_{X,t+1}$ is $\sigma_{X,m}$.

Again, the ε's are conditionally jointly normal and mean zero we have $Cov_t\left(\varepsilon_{a,t+1},\varepsilon^2_{b,t+1}\right)=0$ and $Cov_t\left(\varepsilon_{a,t+1},\varepsilon_{b,t+1}\varepsilon_{c,t+1}\right)=0$ for all a,b,c. Additionally, note that we impose $\sigma_{x,\Lambda}=0$ and that the only non-zero covariance of $\varepsilon_{\Lambda,t+1}$ is $\sigma_{\Lambda,m}$. Thus, the conditional covariance of stock returns with the returns on a long term nominal bond is

$$\begin{aligned}
Cov_t\left(r_{e,t+1},r^{\$}_{n,t+1}\right) &= \beta_{ex}\begin{pmatrix} \left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{x.z} \\ +\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{x,\psi}\end{pmatrix} \\
&+\beta_{eX}B^{\$}_{x,n-1}\sigma^2_X \\
&+\beta_{em}\begin{pmatrix} B^{\$}_{x,n-1}\sigma_{Xm}+B^{\$}_{\lambda,n-1}\sigma_{\Lambda m}+\left(B^{\$}_{z,n-1}+2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right)+C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\sigma_{z,m} \\ +\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)+C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{\psi,m}\end{pmatrix} \\
&+\left[\beta_{ex}\left(2C^{\$}_{z,n-1}\sigma_{xz}\phi_z+C^{\$}_{z\psi,n-1}\sigma_{x\psi}\phi_z\right)+\beta_{em}\left(2C^{\$}_{z,n-1}\sigma_{zm}\phi_z+C^{\$}_{z\psi,n-1}\sigma_{\psi m}\phi_z\right)\right]z_t \\
&+\begin{bmatrix} \beta_{ex}\left(B^{\$}_{x,n-1}\sigma^2_x+B^{\$}_{\lambda,n-1}\sigma_{x,\lambda}+B^{\$}_{\xi,n-1}\sigma_{x,\xi}+C^{\$}_{z\psi,n-1}\sigma_{xz}\phi_\psi+2C^{\$}_{\psi,n-1}\sigma_{x\psi}\phi_\psi\right) \\ +\beta_{em}\left(B^{\$}_{x,n-1}\sigma_{xm}+B^{\$}_{\lambda,n-1}\sigma_{m,\lambda}+B^{\$}_{\xi,n-1}\sigma_{m,\xi}+C^{\$}_{z\psi,n-1}\sigma_{zm}\phi_\psi+2C^{\$}_{\psi,n-1}\sigma_{\psi m}\phi_\psi\right)\end{bmatrix}\psi_t
\end{aligned}$$

Volatility of Real Bond Returns We have

$$r_{n,t+1} - E_t r_{n,t+1} = \left[\begin{array}{c} B_{x,n-1}\psi_t \varepsilon_{x,t+1} + B_{x,n-1}\varepsilon_{X,t+1} + C_{z,n-1}\varepsilon_{z,t+1}^2 + C_{\psi,n-1}\varepsilon_{\psi,t+1}^2 + C_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ + \left(B_{z,n-1} + 2C_{z,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right) + C_{z\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right)\varepsilon_{z,t+1} \\ + \left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{z\psi,n-1}\left(\mu_z\left(1-\phi_z\right) + \phi_z z_t\right)\right)\varepsilon_{\psi,t+1} \end{array} \right]$$

so that

$$\begin{aligned} Var_t\left(r_{n,t+1}\right) &= \left[\begin{array}{c} B_{x,n-1}^2\sigma_X^2 + 2C_{z,n-1}^2 2\sigma_z^4 + 2C_{\psi,n-1}^2\sigma_\psi^4 + C_{z\psi,n-1}^2\left(\sigma_z^2\sigma_\psi^2 + \sigma_{z\psi}^2\right) + \left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2\sigma_z^2 \\ + \left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2\sigma_\psi^2 \\ +2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \times \left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{z,\psi} \end{array} \right] \\ &+ \left[\begin{array}{c} 4\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z,n-1}\phi_z\sigma_z^2 \\ +2\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{z\psi,n-1}\phi_z\sigma_\psi^2 \\ +2\left[\begin{array}{c} 2C_{z,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\ +C_{z\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \end{array} \right]\phi_z\sigma_{z,\psi} \end{array} \right] z_t \\ &+ \left[\begin{array}{c} 2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{z\psi,n-1}\phi_\psi\sigma_z^2 \\ +4\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C_{\psi,n-1}\phi_\psi\sigma_\psi^2 \\ +2\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B_{x,n-1}\sigma_{xz} \\ +2\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B_{x,n-1}\sigma_{x\psi} \\ +2\left[\begin{array}{c} 2C_{\psi,n-1}\left(B_{z,n-1} + 2C_{z,n-1}\mu_z\left(1-\phi_z\right) + C_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ +C_{z\psi,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \end{array} \right]\phi_\psi\sigma_{z,\psi} \end{array} \right]\psi_t \\ &+ \left[4C_{z,n-1}^2\phi_z^2\sigma_z^2 + C_{z\psi,n-1}^2\phi_z^2\sigma_\psi^2 + 4C_{z,n-1}C_{z\psi,n-1}\phi_z^2\sigma_{z,\psi}\right] z_t^2 \\ &+ \left[B_{x,n-1}^2\sigma_x^2 + C_{z\psi,n-1}^2\phi_\psi^2\sigma_z^2 + 4C_{\psi,n-1}^2\phi_\psi^2\sigma_\psi^2 + 2C_{z\psi,n-1}\phi_\psi B_{x,n-1}\sigma_{xz} + 4C_{\psi,n-1}\phi_\psi B_{x,n-1}\sigma_{x\psi} + 4C_{\psi,n-1}C_{z\psi,n-1}\phi_\psi^2\sigma_{z,\psi}\right]\psi_t^2 \\ &+ \left[\begin{array}{c} 4C_{z,n-1}C_{z\psi,n-1}\phi_z\phi_\psi\sigma_z^2 + 4C_{\psi,n-1}\phi_\psi C_{z\psi,n-1}\phi_z\phi_\psi\sigma_\psi^2 + 4C_{z,n-1}\phi_z B_{x,n-1}\sigma_{xz} \\ +2C_{z\psi,n-1}\phi_z B_{x,n-1}\sigma_{x\psi} + 2\left(4C_{z,n-1}C_{\psi,n-1} + C_{z\psi,n-1}^2\right)\sigma_{z\psi}\phi_\psi\phi_z \end{array} \right] z_t\psi_t \end{aligned}$$

Volatility of Nominal Bond Returns We have

$$r^{\$}_{n,t+1} - E_t r^{\$}_{n,t+1} = \left[\begin{array}{c} B^{\$}_{x,n-1}\psi_t \varepsilon_{x,t+1} + B^{\$}_{x,n-1}\varepsilon_{X,t+1} + B^{\$}_{\lambda,n-1}\psi_t \varepsilon_{\lambda,t+1} + B^{\$}_{\lambda,n-1}\varepsilon_{\Lambda,t+1} + B^{\$}_{\xi,n-1}\psi_t \varepsilon_{\xi,t+1} \\ +C^{\$}_{z,n-1}\varepsilon^2_{z,t+1} + C^{\$}_{\psi,n-1}\varepsilon^2_{\psi,t+1} + C^{\$}_{z\psi,n-1}\varepsilon_{z,t+1}\varepsilon_{\psi,t+1} \\ + \left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\left(\mu_z \left(1-\phi_z\right) + \phi_z z_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) \varepsilon_{z,t+1} \\ + \left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C^{\$}_{z\psi,n-1}\left(\mu_z \left(1-\phi_z\right) + \phi_z z_t\right)\right) \varepsilon_{\psi,t+1} \end{array} \right]$$

so that

$$
\begin{aligned}
Var_t\left(r^{\$}_{n,t+1}\right) \;=\; & \left[\begin{array}{c}
B^{\$2}_{x,n-1}\sigma^2_X + B^{\$2}_{\lambda,n-1}\sigma^2_\Lambda + 2C^{\$2}_{z,n-1}\sigma^4_z + 2C^{\$2}_{\psi,n-1}\sigma^4_\psi + C^{\$2}_{z\psi,n-1}\left(\sigma^2_z\sigma^2_\psi + \sigma^2_{z\psi}\right) \\
+\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2\sigma^2_z \\
+\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)^2\sigma^2_\psi \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)\sigma_{z,\psi}
\end{array}\right] \\
& + \left[\begin{array}{c}
4\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z,n-1}\sigma^2_z\phi_z \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{z\psi,n-1}\sigma^2_\psi\phi_z \\
+2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda}\phi_z \\
+2\left[\begin{array}{c}
2C^{\$}_{z,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right) \\
+C^{\$}_{z\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)
\end{array}\right]\sigma_{z,\psi}\phi_z
\end{array}\right] z_t \\
& + \left[\begin{array}{c}
2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{x,n-1}\sigma_{xz} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{x,n-1}\sigma_{x\psi} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{z\psi,n-1}\sigma^2_z\phi_\psi \\
+4\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)C^{\$}_{\psi,n-1}\sigma^2_\psi\phi_\psi \\
+2B^{\$2}_{\lambda,n-1}\sigma_{\lambda,\Lambda} \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\lambda,n-1}\sigma_{z,\lambda} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\lambda,n-1}\sigma_{\psi,\lambda} \\
+2B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\sigma_{\Lambda,\xi} + 4C^{\$}_{\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\Lambda}\phi_\psi \\
+2\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\xi,n-1}\sigma_{\xi,z} \\
+2\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)B^{\$}_{\xi,n-1}\sigma_{\psi,\xi} \\
+2\left[\begin{array}{c}
2C^{\$}_{\psi,n-1}\left(B^{\$}_{z,n-1} + 2C^{\$}_{z,n-1}\mu_z\left(1-\phi_z\right) + C^{\$}_{z\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\
+C^{\$}_{z\psi,n-1}\left(B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right) + C^{\$}_{z\psi,n-1}\mu_z\left(1-\phi_z\right)\right)
\end{array}\right]\sigma_{z,\psi}\phi_\psi
\end{array}\right]\psi_t
\end{aligned}
$$

$$
\begin{aligned}
&+\left[4C^{\$2}_{z,n-1}\phi_z^2\sigma_z^2 + C^{\$2}_{z\psi,n-1}\phi_z^2\sigma_\psi^2 + 4C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\sigma_{z,\psi}\phi_z^2\right] z_t^2 \\
&+\left[\begin{array}{c}
B^{\$2}_{x,n-1}\sigma_x^2 + B^{\$2}_{\lambda,n-1}\sigma_\lambda^2 + B^{\$2}_{\xi,n-1}\sigma_\xi^2 + 2B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}\sigma_{x,\lambda} + 2B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}\sigma_{x,\xi} \\
+2C^{\$}_{z\psi,n-1}B^{\$}_{x,n-1}\sigma_{xz}\phi_\psi + 4C^{\$}_{\psi,n-1}B^{\$}_{x,n-1}\sigma_{x\psi}\phi_\psi + C^{\$2}_{z\psi,n-1}\phi_\psi^2\sigma_z^2 + 4C^{\$2}_{\psi,n-1}\phi_\psi^2\sigma_\psi^2 + 2B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi\lambda} + 2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{z,\lambda}\phi_\psi \\
+4C^{\$}_{\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\lambda}\phi_\psi + 2C^{\$}_{z\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi,z}\phi_\psi + 4C^{\$}_{\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\psi,\xi}\phi_\psi + 4C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\sigma_{z,\psi}\phi_\psi^2
\end{array}\right]\psi_t^2 \\
&+\left[\begin{array}{c}
4C^{\$}_{z,n-1}B^{\$}_{x,n-1}\sigma_{xz}\phi_z + 2C^{\$}_{z\psi,n-1}B^{\$}_{x,n-1}\sigma_{x\psi}\phi_z + 4C^{\$}_{z,n-1}C^{\$}_{z\psi,n-1}\sigma_z^2\phi_z\phi_\psi \\
+4C^{\$}_{\psi,n-1}C^{\$}_{z\psi,n-1}\sigma_\psi^2\phi_z\phi_\psi + 4C^{\$}_{z,n-1}B^{\$}_{\lambda,n-1}\sigma_{z,\lambda}\phi_z + 2C^{\$}_{z\psi,n-1}B^{\$}_{\lambda,n-1}\sigma_{\psi,\lambda}\phi_z \\
+4C^{\$}_{z,n-1}B^{\$}_{\xi,n-1}\sigma_{\xi,z}\phi_z + 2C^{\$}_{z\psi,n-1}B^{\$}_{\xi,n-1}\sigma_{\psi,\xi}\phi_z + 2\left(4C^{\$}_{z,n-1}C^{\$}_{\psi,n-1} + C^{\$2}_{z\psi,n-1}\right)\sigma_{z\psi}\phi_\psi\phi_z
\end{array}\right] z_t\psi_t
\end{aligned}
$$

A.3 Derivations for Constant-z Model

A.3.1 State Variables Processes

The state variables in the constant-z version of the model follow the processes:

$$\begin{aligned} -m_{t+1} &= x_t + \frac{1}{2}\sigma_m^2 + \varepsilon_{m,t+1} \\ x_{t+1} &= \mu_x (1-\phi_x) + \phi_x x_t + \psi_t \varepsilon_{x,t+1} + \varepsilon_{X,t+1} \end{aligned}$$

$$\begin{aligned} \pi_{t+1} &= \lambda_t + \xi_t + \frac{1}{2}\psi_t^2 \sigma_\pi^2 + \psi_t \varepsilon_{\pi,t+1} \\ \lambda_{t+1} &= \lambda_t + \psi_t \varepsilon_{\lambda,t+1} + \varepsilon_{\Lambda,t+1} \\ \xi_{t+1} &= \phi_\xi \xi_t + \psi_t \varepsilon_{\xi,t+1} \\ \psi_{t+1} &= \mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t + \varepsilon_{\psi,t+1} \end{aligned}$$

A.3.2 Pricing Equations

Real Term Structure The price of a single-period zero-coupon real bond satisfies

$$P_{1,t} = E_t \left[\exp\left\{m_{t+1}\right\}\right] = -x_t - \frac{1}{2}\sigma_m^2 + \frac{1}{2}\sigma_m^2 = -x_t$$

1. We conjecture that the price function is exponential affine in x_t and z_t with the form

$$P_{n,t} = \exp\left\{A_n + B_{x,n} x_t + B_{\psi,n}\psi_t + C_{\psi,n}\psi_t^2\right\}.$$

The standard pricing equation implies

$$\begin{aligned} P_{n,t} &= E_t\left[\exp\left\{p_{n-1,t+1} + m_{t+1}\right\}\right] = E_t\left[\exp\left\{A_{n-1} + B_{x,n-1}x_{t+1} + B_{\psi,n-1}\psi_{t+1} + C_{\psi,n-1}\psi_{t+1}^2. - x_t - \frac{1}{2}\sigma_m^2 - \varepsilon_{m,t+1}\right\}\right] \\ &= E_t\left[\exp\left\{\begin{array}{c} A_{n-1} + B_{x,n-1}\left((1-\phi_x)\mu_x + \phi_x x_t + \psi_t\varepsilon_{x,t+1} + \varepsilon_{X,t+1}\right) + B_{\psi,n-1}\left(\left(1-\phi_\psi\right)\mu_\psi + \phi_\psi\psi_t + \varepsilon_{\psi,t+1}\right) \\ + C_{\psi,n-1}\left(\left(1-\phi_\psi\right)\mu_\psi + \phi_\psi\psi_t + \varepsilon_{\psi,t+1}\right)^2. - x_t - \frac{1}{2}\sigma_m^2 - \varepsilon_{m,t+1} \end{array}\right\}\right] \\ &= A_{n-1} + B_{x,n-1}\left((1-\phi_x)\mu_x + \phi_x x_t\right) + B_{\psi,n-1}\left(\left(1-\phi_\psi\right)\mu_\psi + \phi_\psi\psi_t\right) + C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)^2 - x_t - \frac{1}{2}\sigma_m^2 \\ &\quad \times E_t\left[\exp\left\{\mathbf{d}_1'\boldsymbol{\omega}_{t+1} + \boldsymbol{\omega}_{t+1}'\mathbf{D}_2\boldsymbol{\omega}_{t+1}\right\}\right] \end{aligned}$$

where $\boldsymbol{\omega}'_{t+1} = (\varepsilon_{X,t+1}, \varepsilon_{m,t+1}, \varepsilon_{x,t+1}, \varepsilon_{\psi,t+1})\, \tilde{}\, N(0, \boldsymbol{\Sigma}_\omega)$,

$$\mathbf{d}_1 = \begin{pmatrix} B_{x,n-1} \\ -1 \\ B_{x,n-1}\psi_t \\ B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) \end{pmatrix}$$

$$\mathbf{D}_2 = \begin{pmatrix} 0 & & 0 \\ & & \\ 0 & & C_{\psi,n-1} \end{pmatrix}$$

Following Campbell, Chan, and Viceira (2003), we complete the square to calculate

$$\begin{aligned} E_t\left[\exp\left\{\mathbf{d}'_1\boldsymbol{\omega}_{t+1} + \boldsymbol{\omega}'_{t+1}\mathbf{D}_2\boldsymbol{\omega}_{t+1}\right\}\right] &= \frac{\left|\boldsymbol{\Sigma}_\omega\right|^{-1/2}}{\left|\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right|^{1/2}} \exp\left\{\frac{1}{2}\mathbf{d}_1\left(\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right)^{-1}\mathbf{d}'_1\right\} \\ &= \exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}_\omega\right| + \frac{1}{2}\log\left|\mathbf{G}\right| + \frac{1}{2}\mathbf{d}_1\mathbf{G}\mathbf{d}'_1\right\} \end{aligned}$$

where $\mathbf{G} = \left(\boldsymbol{\Sigma}_\omega^{-1} - 2\mathbf{D}_2\right)^{-1}$. Let g_{ij} be the ij-th element of $\mathbf{G}$. Then expanding and collecting terms gives

$$\begin{aligned} p_{n,t} &= \left[\begin{array}{c} A_{n-1} + B_{x,n-1}\left(\left(1-\phi_x\right)\mu_x + \phi_x x_t\right) + B_{\psi,n-1}\left(\left(1-\phi_\psi\right)\mu_\psi + \phi_\psi\psi_t\right) + C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)^2 \\ -x_t - \frac{1}{2}\sigma_m^2 - \frac{1}{2}\log\left|\boldsymbol{\Sigma}_\omega\right| + \frac{1}{2}\log\left|\mathbf{G}\right| + \frac{1}{2}g_{11}B_{x,n-1}^2 + \frac{1}{2}g_{22} + \frac{1}{2}g_{33}B_{x,n-1}^2\psi_t^2 \\ +\frac{1}{2}g_{44}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right)^2 \\ -g_{12}B_{x,n-1} + g_{13}B_{x,n-1}^2\psi_t + g_{14}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right) \\ -g_{23}B_{x,n-1}\psi_t - g_{24}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right) \\ +g_{34}B_{x,n-1}\psi_t\left(B_{\psi,n-1} + 2C_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)\right) \end{array}\right] \\ &= \left[\begin{array}{c} A_{n-1} + B_{x,n-1}\left(1-\phi_x\right)\mu_x + B_{x,n-1}\phi_x x_t + B_{\psi,n-1}\left(1-\phi_\psi\right)\mu_\psi + B_{\psi,n-1}\phi_\psi\psi_t \\ +C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi\psi_t + C_{\psi,n-1}\phi_\psi^2\psi_t^2 \\ -x_t - \frac{1}{2}\sigma_m^2 - \frac{1}{2}\log\left|\boldsymbol{\Sigma}_\omega\right| + \frac{1}{2}\log\left|\mathbf{G}\right| + \frac{1}{2}g_{11}B_{x,n-1}^2 + \frac{1}{2}g_{22} + \frac{1}{2}g_{33}B_{x,n-1}^2\psi_t^2 \\ +\frac{1}{2}g_{44}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 + 2g_{44}C_{\psi,n-1}^2\phi_\psi^2\psi_t^2 + 2g_{44}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{\psi,n-1}\phi_\psi\psi_t \\ -g_{12}B_{x,n-1} + g_{13}B_{x,n-1}^2\psi_t + g_{14}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + 2g_{14}B_{x,n-1}C_{\psi,n-1}\phi_\psi\psi_t \\ -g_{23}B_{x,n-1}\psi_t - g_{24}\left(B_{\psi,n-1} + 2g_{24}C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) - 2g_{24}C_{\psi,n-1}\phi_\psi\psi_t \\ +g_{34}B_{x,n-1}\psi_t\left(B_{\psi,n-1} + 2B_{x,n-1}C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) + 2g_{34}B_{x,n-1}C_{\psi,n-1}\phi_\psi\psi_t^2 \end{array}\right] \end{aligned}$$

Expanding and collecting terms yields

$$
\begin{aligned}
A_n &= \left[\begin{array}{c} A_{n-1} + B_{x,n-1}(1-\phi_x)\mu_x + B_{\psi,n-1}\left(1-\phi_\psi\right)\mu_\psi + C_{\psi,n-1}\mu_\psi^2\left(1-\phi_\psi\right)^2 \\ -\frac{1}{2}\sigma_m^2 - \frac{1}{2}\log|\mathbf{\Sigma}_\omega| + \frac{1}{2}\log|\mathbf{G}| + \frac{1}{2}g_{11}B_{x,n-1}^2 + \frac{1}{2}g_{22} \\ +\frac{1}{2}g_{44}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)^2 - g_{12}B_{x,n-1} + g_{14}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \\ -g_{24}\left(B_{\psi,n-1} + 2g_{24}C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \end{array} \right] \\
B_{x,n} &= B_{x,n-1}\phi_x - 1 \\
B_{\psi,n} &= \left[\begin{array}{c} B_{\psi,n-1}\phi_\psi + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\phi_\psi + 2g_{44}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C_{\psi,n-1}\phi_\psi \\ +g_{13}B_{x,n-1}^2 + 2g_{14}B_{x,n-1}C_{\psi,n-1}\phi_\psi - g_{23}B_{x,n-1} - 2g_{24}C_{\psi,n-1}\phi_\psi \\ +g_{34}B_{x,n-1}\left(B_{\psi,n-1} + 2C_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right) \end{array} \right] \\
C_{\psi,n} &= \left[C_{\psi,n-1}\phi_\psi^2 + \frac{1}{2}g_{33}B_{x,n-1}^2 + 2g_{44}C_{\psi,n-1}^2\phi_\psi^2 + 2g4B_{x,n-1}C_{\psi,n-1}\phi_\psi \right]
\end{aligned}
$$

Nominal Term Structure The price of a single-period zero-coupon nominal bond satisfies

$$
\begin{aligned}
P_{1,t}^{\$} &= E_t\left[\exp\left\{m_{t+1} - \pi_{t+1}\right\}\right] \\
&= E_t\left[\exp\left\{-x_t - \frac{1}{2}\sigma_m^2 - \varepsilon_{m,t+1} - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2\sigma_\pi^2 - \psi_t\varepsilon_{\pi,t+1}\right\}\right] \\
&= \exp\left\{-x_t - \frac{1}{2}\sigma_m^2 - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2\sigma_\pi^2 + \frac{1}{2}\sigma_m^2 + \frac{1}{2}\psi_t^2\sigma_\pi^2 + \psi_t\sigma_{m\pi}\right\} \\
&= \exp\left\{-x_t - \lambda_t - \xi_t + \psi_t\sigma_{m\pi}\right\}
\end{aligned}
$$

where the last equality follows from the joint conditional normality of $z_t\varepsilon_{m,t+1}$ and $\psi_t\varepsilon_{\pi,t+1}$.

We now guess that the price function is exponential linear-quadratic in the state variables with the following form:

$$
P_{n,t}^{\$} = \exp\left\{A_n^{\$} + B_{x,n}^{\$}x_t + B_{\lambda,n}^{\$}\lambda_t + B_{\xi,n}^{\$}\xi_t + B_{\psi,n}^{\$}\psi_t + C_{\psi,n}^{\$}\psi_t^2\right\}
$$

The standard pricing equation then implies

$$
\begin{aligned}
P_{n,t}^{\$} &= E_t\left[\exp\left\{p_{n-1,t+1}^{\$} + m_{t+1} - \pi_{t+1}\right\}\right] \\
&= E_t\left[\exp\left\{ \begin{array}{c} A_{n-1}^{\$} + B_{x,n-1}^{\$}x_{t+1} + B_{\lambda,n-1}^{\$}\lambda_{t+1} + B_{\xi,n-1}^{\$}\xi_{t+1} + B_{\psi,n-1}^{\$}\psi_{t+1} + C_{\psi,n-1}^{\$}\psi_{t+1}^2 \\ -x_t - \frac{1}{2}\sigma_m^2 - \varepsilon_{m,t+1} - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2\sigma_\pi^2 - \psi_t\varepsilon_{\pi,t+1} \end{array} \right\}\right] \\
&= \exp\left\{ \begin{array}{c} A_{n-1}^{\$} + B_{x,n-1}^{\$}\left(\mu_x(1-\phi_x) + \phi_x x_t\right) + B_{\lambda,n-1}^{\$}\left(\mu_\lambda + \lambda_t\right) + B_{\xi,n-1}^{\$}\phi_\xi\xi_t \\ +B_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) + C_{\psi,n-1}^{\$}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right)^2 \\ -x_t - \frac{1}{2}\sigma_m^2 - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2\sigma_\pi^2 \end{array} \right\} \\
&\quad \times E_t\left[\exp\left\{\mathbf{d}_1^{\$\prime}\boldsymbol{\omega}_{t+1}^{\$} + \boldsymbol{\omega}_{t+1}^{\$\prime}\mathbf{D}_2^{\$}\boldsymbol{\omega}_{t+1}^{\$}\right\}\right]
\end{aligned}
$$

where $\boldsymbol{\omega}^{\$\prime}_{t+1} = (\varepsilon_{X,t+1}, \varepsilon_{\Lambda,t+1}, \varepsilon_{\lambda,t+1}, \varepsilon_{m,t+1}, \varepsilon_{\pi,t+1}, \varepsilon_{x,t+1}, \varepsilon_{\xi,t+1}, \varepsilon_{\psi,t+1}) \,\tilde{}\, N\left(0, \boldsymbol{\Sigma}^{\$}_{\omega}\right)$,

$$\mathbf{d}^{\$}_1 = \begin{pmatrix} B^{\$}_{x,n-1} \\ B^{\$}_{\lambda,n-1} \\ B^{\$}_{\lambda,n-1}\psi_t \\ -1 \\ -\psi_t \\ B^{\$}_{x,n-1}\psi_t \\ B^{\$}_{\xi,n-1}\psi_t \\ B^{\$}_{\psi,n-1} + 2C^{\$}_{\psi,n-1}\left(\mu_\psi\left(1-\phi_\psi\right) + \phi_\psi\psi_t\right) \end{pmatrix}$$

$$\mathbf{D}^{\$}_2 = \begin{pmatrix} 0 & \cdots & 0 \\ & & \vdots \\ \vdots & \ddots & \\ 0 & \cdots & C^{\$}_{\psi,n-1} \end{pmatrix}$$

Following Campbell, Chan, and Viceira (2003), we complete the square to calculate

$$\begin{aligned} E_t\left[\exp\left\{\mathbf{d}^{\$\prime}_1\boldsymbol{\omega}^{\$}_{t+1} + \boldsymbol{\omega}^{\$\prime}_{t+1}\mathbf{D}^{\$}_2\boldsymbol{\omega}^{\$}_{t+1}\right\}\right] &= \frac{\left|\boldsymbol{\Sigma}^{\$}_{\omega}\right|^{-1/2}}{\left|\boldsymbol{\Sigma}^{\$-1}_{\omega} - 2\mathbf{D}^{\$}_2\right|^{1/2}} \exp\left\{\frac{1}{2}\mathbf{d}^{\$}_1\left(\boldsymbol{\Sigma}^{\$-1}_{\omega} - 2\mathbf{D}^{\$}_2\right)^{-1}\mathbf{d}^{\$\prime}_1\right\} \\ &= \exp\left\{-\frac{1}{2}\log\left|\boldsymbol{\Sigma}^{\$}_{\omega}\right| + \frac{1}{2}\log\left|\mathbf{G}^{\$}\right| + \frac{1}{2}\mathbf{d}^{\$}_1\mathbf{G}^{\$}\mathbf{d}^{\$\prime}_1\right\} \end{aligned}$$

where $\mathbf{G}^{\$} = \left(\boldsymbol{\Sigma}^{\$-1}_{\omega} - 2\mathbf{D}^{\$}_2\right)^{-1}$. Let $g^{\$}_{ij}$ be the ij-th element of $\mathbf{G}$. Then expanding and collecting terms gives $g^{\$}$

$$p_{n,t}^{\$} = \left[\begin{array}{c} A_{n-1}^{\$} + B_{x,n-1}^{\$} \left(\mu_x \left(1-\phi_x\right) + \phi_x x_t\right) + B_{\lambda,n-1}^{\$} \lambda_t + B_{\xi,n-1}^{\$} \phi_\xi \xi_t \\ + B_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right) + C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)^2 \\ -x_t - \frac{1}{2}\sigma_m^2 - \lambda_t - \xi_t - \frac{1}{2}\psi_t^2 \sigma_\pi^2 - \frac{1}{2}\log\left|\mathbf{\Sigma}_\omega^{\$}\right| + \frac{1}{2}\log\left|\mathbf{G}^{\$}\right| \\ + \frac{1}{2} g_{11}^{\$} B_{x,n-1}^{\$2} + \frac{1}{2} g_{22}^{\$} B_{\lambda,n-1}^{\$2} + \frac{1}{2} g_{33}^{\$} B_{\lambda,n-1}^{\$2} \psi_t^2 + \frac{1}{2} g_{44}^{\$} + \frac{1}{2} g_{55}^{\$} \psi_t^2 + \frac{1}{2} g_{66}^{\$} B_{x,n-1}^{\$2} \psi_t^2 + \frac{1}{2} g_{77}^{\$} B_{\xi,n-1}^{\$2} \psi_t^2 \\ + \frac{1}{2} g_{88}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right)^2 \\ + g_{12} B_{x,n-1}^{\$} B_{\lambda,n-1}^{\$} + g_{13} B_{x,n-1}^{\$} B_{\lambda,n-1}^{\$} \psi_t - g_{14} B_{x,n-1}^{\$} - g_{15} B_{x,n-1}^{\$} \psi_t + g_{16} B_{x,n-1}^{\$2} \psi_t + g_{17} B_{x,n-1}^{\$} B_{\xi,n-1}^{\$} \psi_t \\ + g_{18} B_{x,n-1}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) \\ + g_{23}^{\$} B_{\lambda,n-1}^{\$2} \psi_t - g_{24}^{\$} B_{\lambda,n-1}^{\$} - g_{25}^{\$} B_{\lambda,n-1}^{\$} \psi_t + g_{26}^{\$} B_{\lambda,n-1}^{\$} B_{x,n-1}^{\$} \psi_t + g_{27}^{\$} B_{\lambda,n-1}^{\$} B_{\xi,n-1}^{\$} \psi_t \\ + g_{28}^{\$} B_{\lambda,n-1}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) \\ - g_{34}^{\$} B_{\lambda,n-1}^{\$} \psi_t - g_{35}^{\$} B_{\lambda,n-1}^{\$} \psi_t^2 + g_{36}^{\$} B_{\lambda,n-1}^{\$} B_{x,n-1}^{\$} \psi_t^2 + g_{37}^{\$} B_{\lambda,n-1}^{\$} B_{\xi,n-1}^{\$} \psi_t^2 \\ + g_{38}^{\$} B_{\lambda,n-1}^{\$} \psi_t \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) \\ + g_{45}^{\$} \psi_t - g_{46}^{\$} B_{x,n-1}^{\$} \psi_t - g_{47}^{\$} B_{\xi,n-1}^{\$} \psi_t - g_{48}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) \\ - g_{56}^{\$} B_{x,n-1}^{\$} \psi_t^2 - g_{57}^{\$} B_{\xi,n-1}^{\$} \psi_t^2 - g_{58}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) \psi_t \\ + g_{67}^{\$} B_{x,n-1}^{\$} B_{\xi,n-1}^{\$} \psi_t^2 \\ + g_{68}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) B_{x,n-1}^{\$} \psi_t \\ + g_{78}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \left(\mu_\psi \left(1-\phi_\psi\right) + \phi_\psi \psi_t\right)\right) B_{\xi,n-1}^{\$} \psi_t \end{array} \right]$$

Thus, the coefficients of the pricing equation satisfy

$$\begin{aligned} A_n^{\$} &= \left[\begin{array}{c} A_{n-1}^{\$} + B_{x,n-1}^{\$} \mu_x \left(1-\phi_x\right) + B_{\psi,n-1}^{\$} \mu_\psi \left(1-\phi_\psi\right) + C_{\psi,n-1}^{\$} \mu_\psi^2 \left(1-\phi_\psi\right)^2 - \frac{1}{2}\sigma_m^2 - \frac{1}{2}\log\left|\mathbf{\Sigma}_\omega^{\$}\right| + \frac{1}{2}\log\left|\mathbf{G}^{\$}\right| \\ + \frac{1}{2} g_{11}^{\$} B_{x,n-1}^{\$2} + \frac{1}{2} g_{22}^{\$} B_{\lambda,n-1}^{\$2} + \frac{1}{2} g_{44}^{\$} + \frac{1}{2} g_{88}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \mu_\psi \left(1-\phi_\psi\right)\right)^2 \\ + g_{12} B_{x,n-1}^{\$} B_{\lambda,n-1}^{\$} - g_{14} B_{x,n-1}^{\$} + g_{18} B_{x,n-1}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \mu_\psi \left(1-\phi_\psi\right)\right) \\ - g_{24}^{\$} B_{\lambda,n-1}^{\$} + g_{28}^{\$} B_{\lambda,n-1}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \mu_\psi \left(1-\phi_\psi\right)\right) - g_{48}^{\$} \left(B_{\psi,n-1}^{\$} + 2C_{\psi,n-1}^{\$} \mu_\psi \left(1-\phi_\psi\right)\right) \end{array} \right] \\ B_{x,n}^{\$} &= B_{x,n-1}^{\$} \phi_x - 1 \\ B_{\lambda,n}^{\$} &= B_{\lambda,n-1}^{\$} - 1 \\ B_{\xi,n}^{\$} &= B_{\xi,n-1}^{\$} \phi_\xi - 1 \end{aligned}$$

$$B^{\$}_{\psi,n}=\left[\begin{array}{c}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\phi_\psi+2g^{\$}_{88}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)C^{\$}_{\psi,n-1}\phi_\psi\\ +g_{13}B^{\$}_{x,n-1}B^{\$}_{\lambda,n-1}-g_{15}B^{\$}_{x,n-1}+g_{16}B^{\$2}_{x,n-1}+g_{17}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}+2g_{18}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_\psi\\ g^{\$}_{23}B^{\$2}_{\lambda,n-1}-g^{\$}_{25}B^{\$}_{\lambda,n-1}+g^{\$}_{26}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1}+g^{\$}_{27}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}+2g^{\$}_{28}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_\psi\\ -g^{\$}_{34}B^{\$}_{\lambda,n-1}+g^{\$}_{38}B^{\$}_{\lambda,n-1}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)\\ +g^{\$}_{45}-g^{\$}_{46}B^{\$}_{x,n-1}-g^{\$}_{47}B^{\$}_{\xi,n-1}-2g^{\$}_{48}C^{\$}_{\psi,n-1}\phi_\psi\\ -g^{\$}_{59}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)+g^{\$}_{68}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{x,n-1}\\ +g^{\$}_{78}\left(B^{\$}_{\psi,n-1}+2C^{\$}_{\psi,n-1}\mu_\psi\left(1-\phi_\psi\right)\right)B^{\$}_{\xi,n-1}\end{array}\right]$$

$$C^{\$}_{\psi,n}=\left[\begin{array}{c}C^{\$}_{\psi,n-1}\phi^2_\psi-\frac{1}{2}\sigma^2_\pi+\frac{1}{2}g^{\$}_{33}B^{\$2}_{\lambda,n-1}+\frac{1}{2}g^{\$}_{55}+\frac{1}{2}g^{\$}_{66}B^{\$2}_{x,n-1}+\frac{1}{2}g^{\$}_{77}B^{\$2}_{\xi,n-1}+2g^{\$}_{88}C^{\$2}_{\psi,n-1}\phi^2_\psi\\ -g^{\$}_{35}B^{\$}_{\lambda,n-1}+g^{\$}_{36}B^{\$}_{\lambda,n-1}B^{\$}_{x,n-1}+g^{\$}_{37}B^{\$}_{\lambda,n-1}B^{\$}_{\xi,n-1}+2g^{\$}_{38}B^{\$}_{\lambda,n-1}C^{\$}_{\psi,n-1}\phi_\psi\\ -g^{\$}_{56}B^{\$}_{x,n-1}-g^{\$}_{57}B^{\$}_{\xi,n-1}-2g^{\$}_{58}C^{\$}_{\psi,n-1}\phi_\psi+g^{\$}_{67}B^{\$}_{x,n-1}B^{\$}_{\xi,n-1}+2g^{\$}_{68}B^{\$}_{x,n-1}C^{\$}_{\psi,n-1}\phi_\psi\\ +2g^{\$}_{78}B^{\$}_{\xi,n-1}C^{\$}_{\psi,n-1}\phi_\psi\end{array}\right]$$

where $B^{\$}_{x,1}=-1, B^{\$}_{\lambda,1}=-1$, $B^{\$}_{\xi,1}=-1$, $B^{\$}_{\psi,1}=\sigma_{m\pi}$ and all other coefficients are zero at $n=1$.